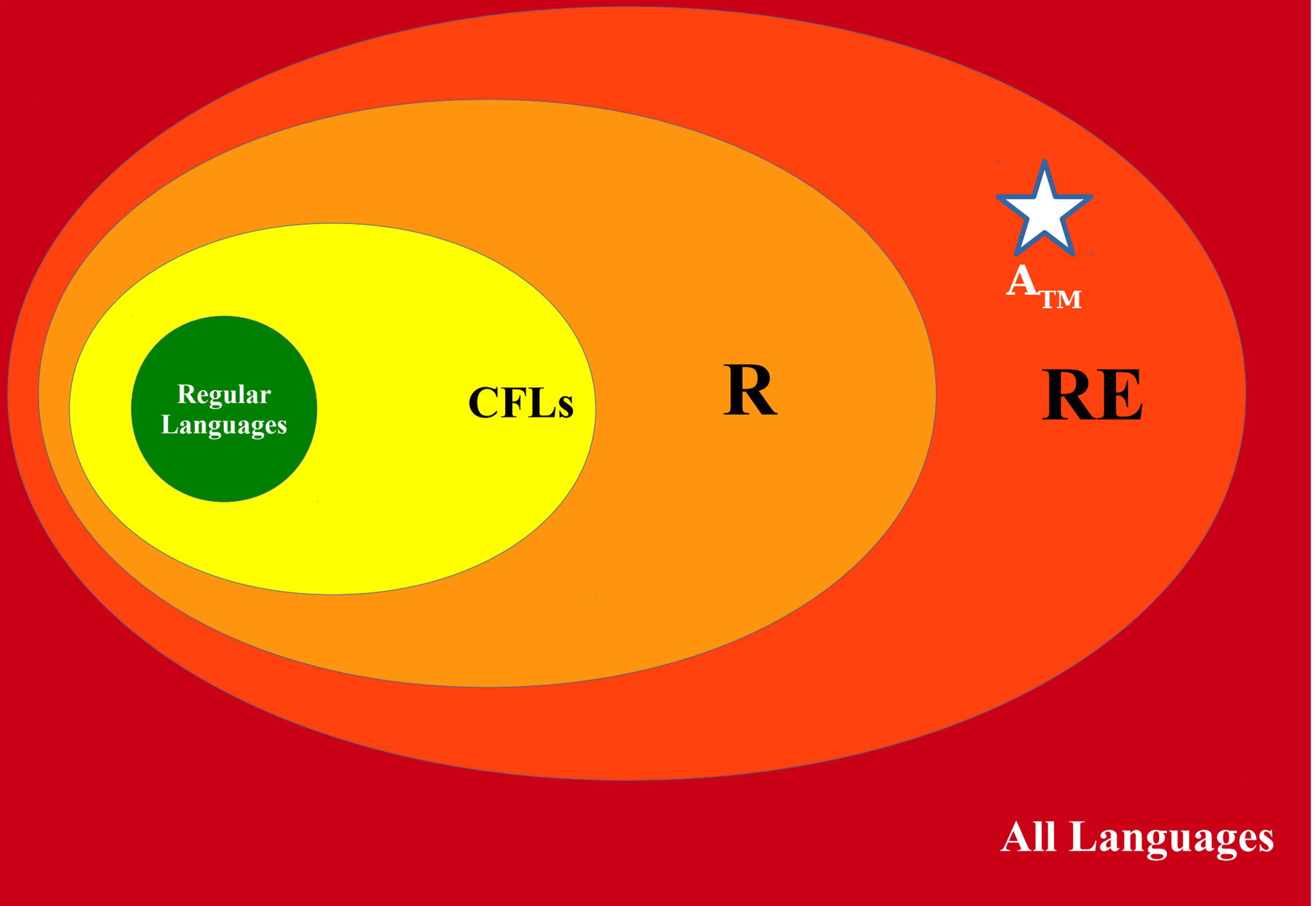
The Class RE


Regular Languages
CFLs
R
RE
A_{TM}
All Languages

More Undecidability Results

The Halting Problem

- The most famous undecidable problem is the ***halting problem***, which asks:

  Given a TM *M* and a string *w*,
  will *M* halt* when run on *w*?

- As a formal language, this problem would be expressed as

  ***HALT* = { ⟨*M*, *w*⟩ | *M* is a TM that halts on *w* }**

- How hard is this problem to solve?

* i.e., accept or reject (“halting” execution, as opposed to infinite loop)

$HALT \in \mathbf{RE}$

- ***Claim:*** $HALT \in \mathbf{RE}$.
- ***Idea:*** If you were certain that a TM M halted on a string w, could you convince me of that?
- Yes – just run M on w and see what happens!

```
bool checkHalt(TM M, string w) {
   // This might infinite loop, and if it does, we will
   // not reach the "if" code below this
   bool result = M(w);

   // whether result is true or false, at least M did halt,
   // so we return true
   if (result || !result) {
      return true;
   }
}
```

$HALT$ and A_{TM}

- Comparing recognizer TMs for $HALT$ and A_{TM}

```
bool checkHalt(TM M, string w) {
   // This might infinite loop
   bool result = M(w);

   // Accept both true and false
   if (result || !result) {
      return true;
   }
}
```

```
bool checkATM(TM M, string w) {
   // This might infinite loop
   bool result = M(w);

   // Accept only true
   if (result) {
      return true;
   } else {
      return false;
   }
}
```

$HALT \notin \mathbf{R}$

- ***Claim:*** $HALT \notin \mathbf{R}$.
- If *HALT* is decidable, there would exist some *decider* function

  `bool willHalt(TM M, string w)`

  that reports whether the program `M` will halt when run on the given input `w`.
- Then, we could do the same trickster setup we saw for A_{TM}...

```
bool trickster(string input) {
   string me = mySource();
   if (willHalt(me, input)) {
      while (true) {
         // loop infinitely
      }
   } else {
      return true;
   }
}
```

Theorem: *HALT* $\notin$ **R**.

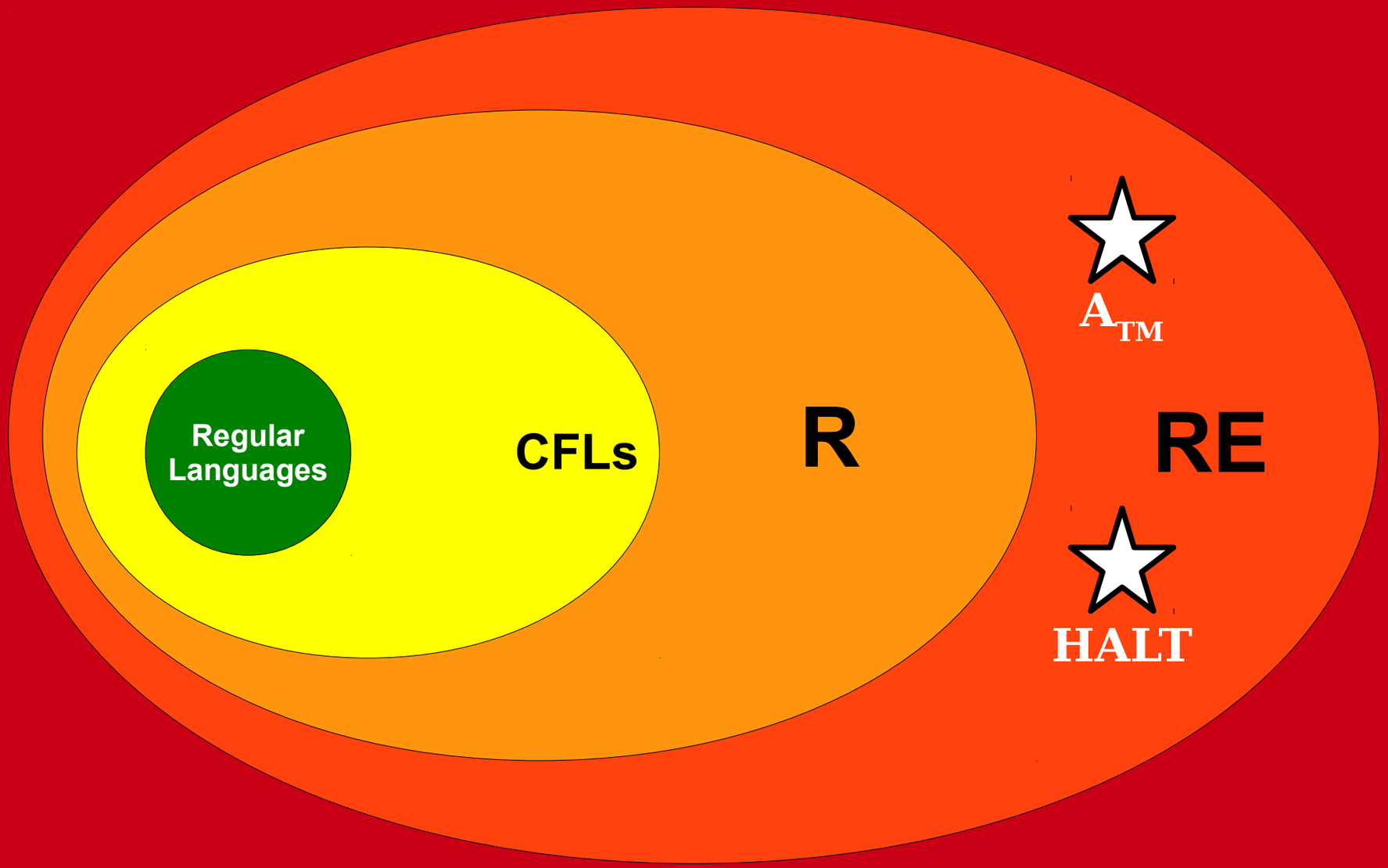
Proof: By contradiction; assume that *HALT* $\in$ **R**. Then there's a decider *D* for *HALT*, which we can represent in software as a method `willHalt` that takes as input the source code of a program and an input, then returns true if the program halts on the input and false otherwise.

Given this, we could then construct this program *Trickster*:

```
bool trickster(string input) {
    string me = mySource();
    if (willHalt(me, input)) {
        while (true) {
            // loop infinitely
        }
    } else {
        return true;
    }
}
```

Choose any string *w* and trace through the execution of program *Trickster* on input *w*, focusing on the answer given back by the `willHalt` method. If `willHalt(me, input)` returns true, then *Trickster* must halt on its input *w*. However, in this case *Trickster* proceeds to loop infinitely on *w*. Otherwise, if `willHalt(me, input)` returns false, then *Trickster* must not halt its input *w*. However, in this case *Trickster* proceeds to accept its input *w*.

In both cases we reach a contradiction, so our assumption must have been wrong. Therefore, *HALT* $\notin$ **R**. ■


Regular Languages
CFLs
R
RE
A_{TM}
HALT
All Languages

The Class RE

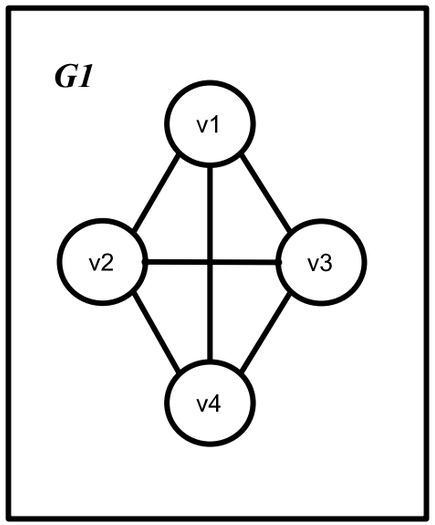
- Languages L that are in RE, *but not* R, are those where:
  - We can build a TM M, where $\mathscr{L}(M) = L$
  - That TM M has the risk of getting stuck in an infinite loop for at least some input string(s)
    - But by definition of $\mathscr{L}(M)$, only input strings that are not in L are at risk of looping in M
- Just like the class Regular was defined in multiple ways (DFAs, NFAs, RegExes), today we'll learn **another way** to define this class RE!

Get ready to answer
some questions in rapid-fire style!
(only about 4 seconds per question)

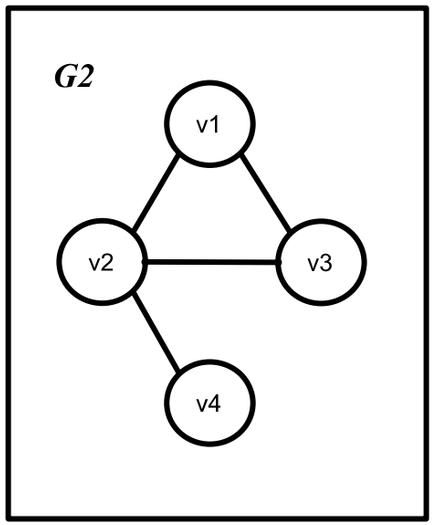
Definition:

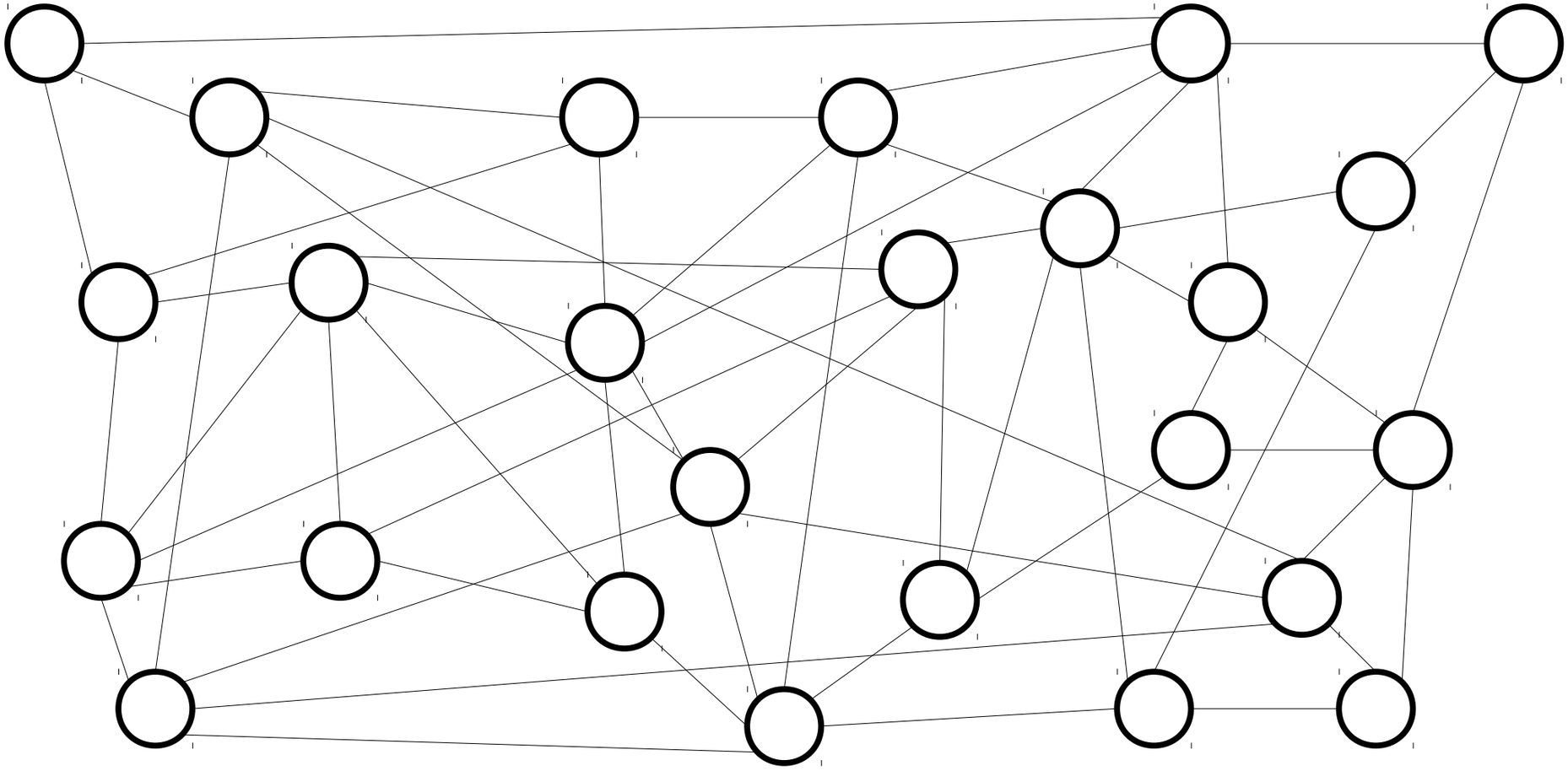
A ***k*-Clique** is a set of k vertices of a graph that are all adjacent to each other (all possible edges between those k vertices are present in the graph).

has a 4-Clique:



does *not* have a 4-Clique (has a 3-Clique though):

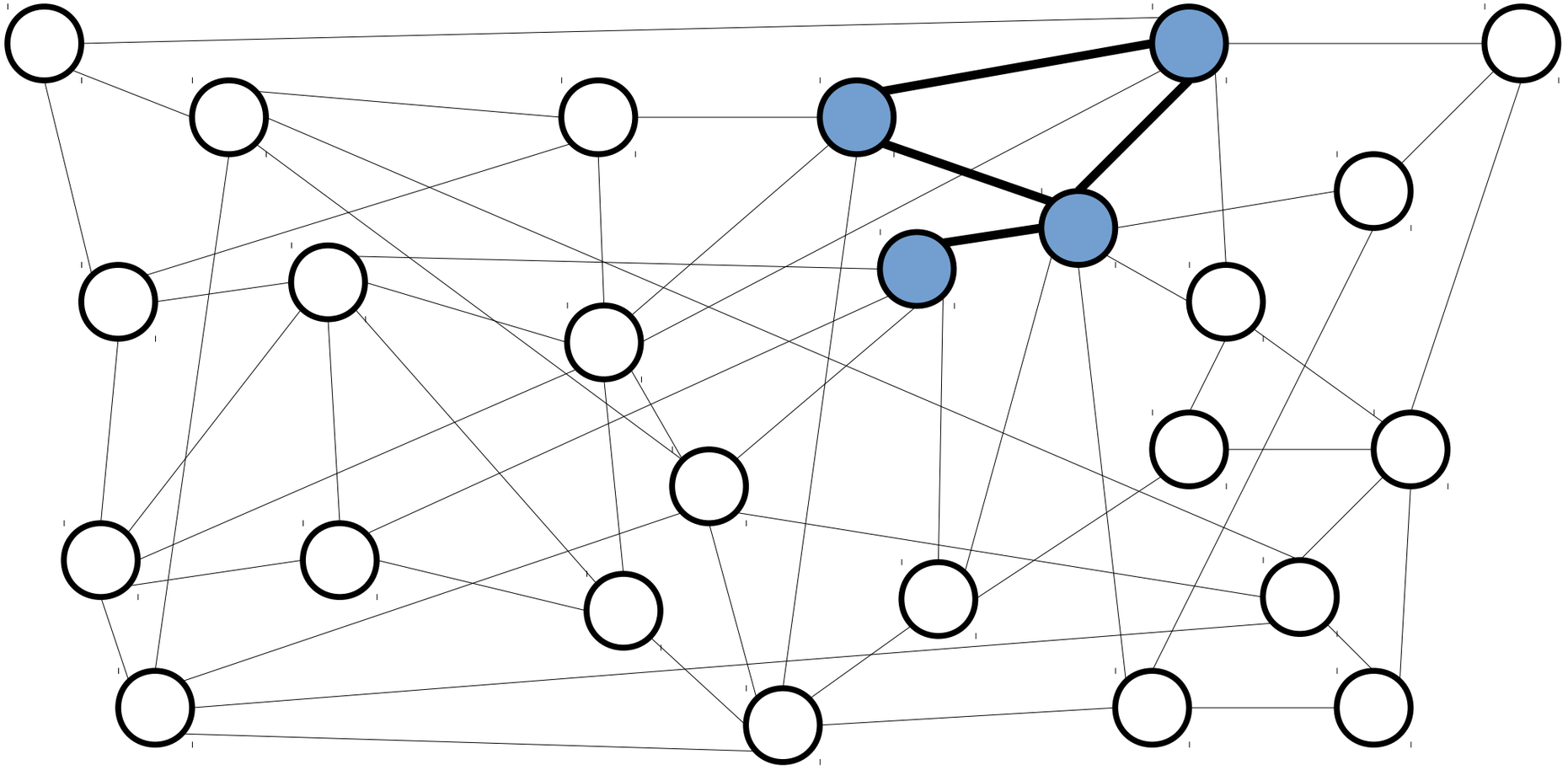




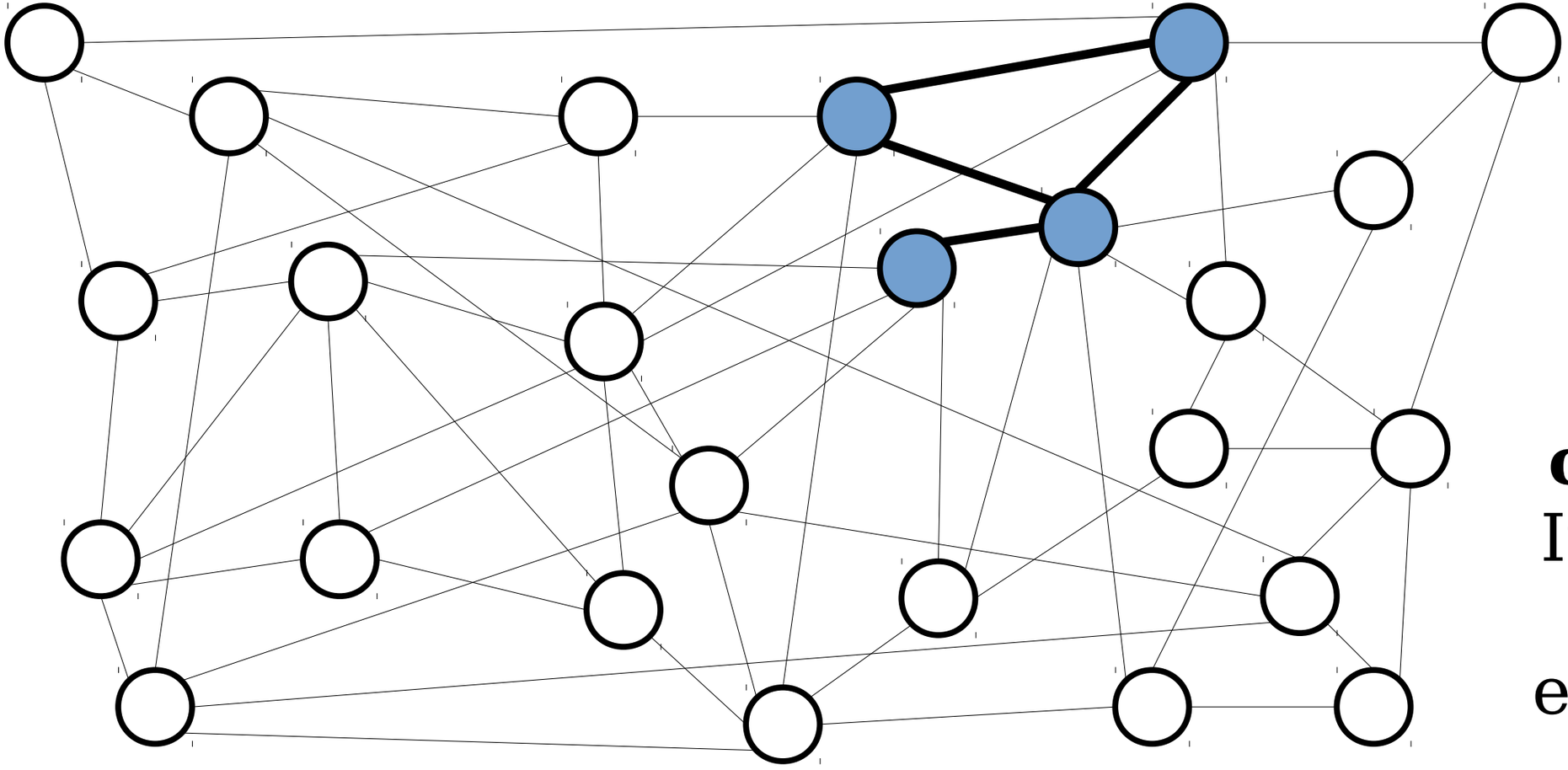
QUICK REACTION: Does this graph contain a 4-clique?

Reflection:

Hm, that was kind of hard to assess in just 4 seconds! What if I select and highlight just some of the nodes for you, would that be a helpful hint?



WITH A “HINT”(?): Does this graph contain a 4-clique?

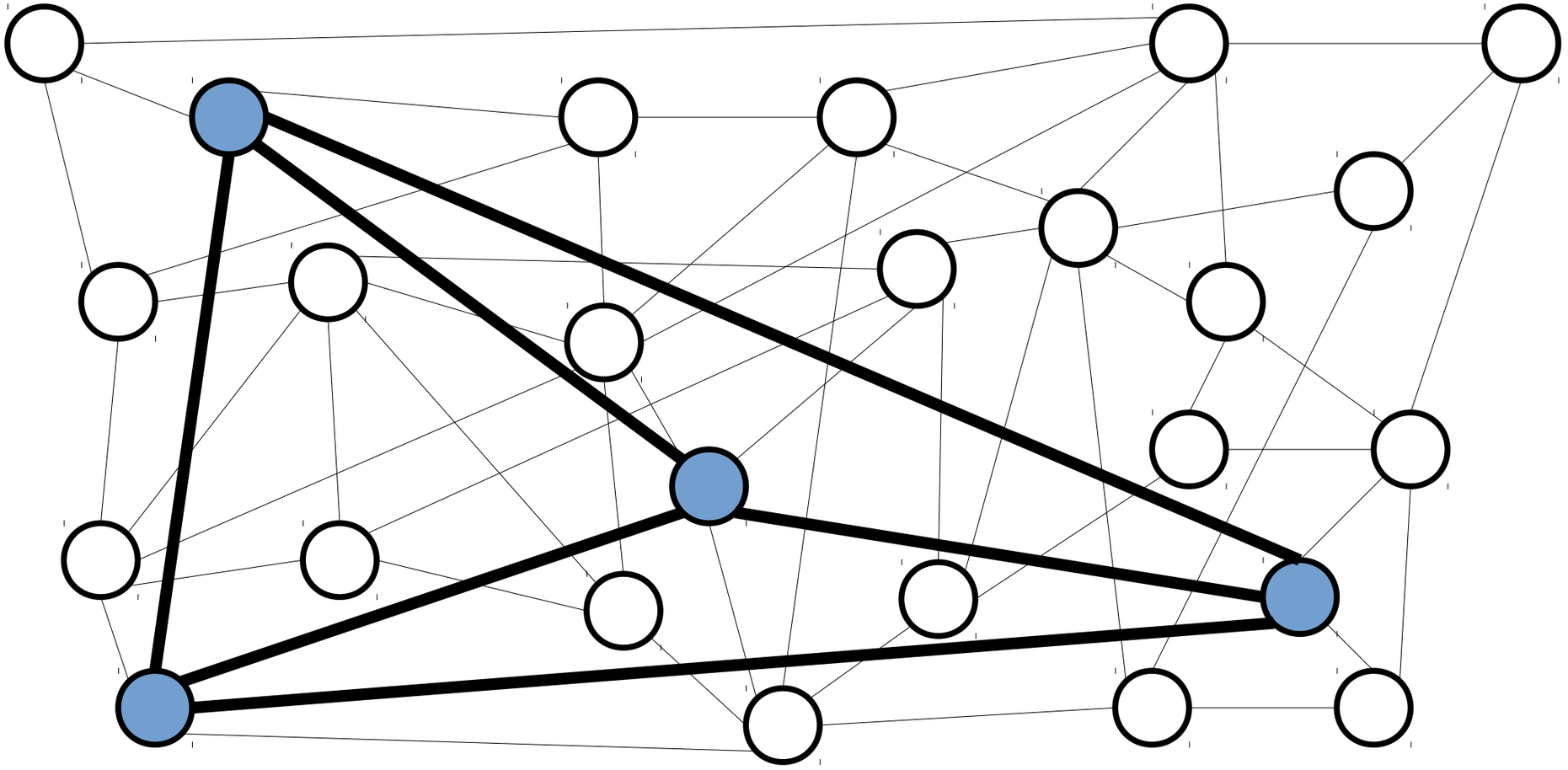


WITH A “HINT”(?): Does this graph contain a 4-clique?

Reflection:

That was a terrible so-called “hint”!

It didn’t make the problem any easier to solve. :-(



WITH A *NEW* HINT: Does this graph contain a 4-clique?

Reflection:

The hint **format** (highlight some subset of 4 nodes) was a good format, but the hint **contents** are only really helpful if they are the correct subset.

Discussion Question:

We found an effective, concise hint format for proving that a graph has a 4-Clique.

What about for proving a graph does **not** have a 4-Clique? What would an effective, concise hint format for that look like?

Reflection:

Highlighting some subset of 4 nodes is **not** a good “hint” format for proving a given graph does *not* have a 4-clique. And in fact, there isn’t any concise *(we’ll define that more rigorously in a second)* format that would work for that. It’s inherently hard to prove a negative.

Key intuition behind our next way of defining RE:

A language L is in **RE** if, for any string w, **<u>if</u>** you know that $w \in L$, then there is some piece of evidence (a "hint") you could provide to make the problem of checking the fact that $w \in L$ very easy.

(But it may not be similarly feasible to present some "hint" that makes the problem of checking that $w \notin L$ very easy.)

Remarks on our Graph Clique example

The problem of saying whether a given Graph contains a **4-clique** **is in *R***. You may be able to solve it *faster* with a hint *(which makes it a fun illustration of the principle of "hints" to use in class)*, but a TM can do it in finite time without a hint.

For the purposes of deciding whether a language is *RE*, speed doesn't really matter, as long as the time is finite.

What that does mean is that the hint has to be finite size, and **"finite" is our definition of "concise" hint**.

More examples of helpful hints vs unhelpful hints

Verification

```
/* imagine 5000 lines of TM code */
```

Does this code HALT on input
“abbababababbbb”?

Verification

```
/* imagine 5000 lines of TM code */
```

Does this code HALT on input “abbababababbbb”?

Reflection:

We know that we are not able to answer this in the general case without the possibility of looping (*HALT* in RE, not R). Is there a finite-length “hint” format we could use to help us decide this?

Verification

```
/* imagine 5000 lines of TM code */
```

Does this code HALT on input
“abbababababbbb”?

Stay with the idea of trying to solve this by running the TM on the input as a test. For input strings that ***do*** HALT in this TM, what hint could help us do that safely (no infinite loop)?

Reflection:

We know that we are not able to answer this in the general case without the possibility of looping (*HALT* in RE, not R). And in particular, if you try to solve it by just running the TM on the input as a test, it might infinite loop. Is there a finite-length “hint” format we could use to help us decide this?

Verification

```
/* imagine 5000 lines of TM code */
```

Does this code HALT on input
“abbababababbbb”?

Stay with the idea of trying to solve this by running the TM on the input as a test. For input strings that ***do*** HALT in this TM, what hint could help us do that safely (no infinite loop)?

Plan:

Use a hint format of “the number of steps to run the TM to observe it halting on this input string.” We will run that many steps and if we see halting, great!

* finite-length
* there is a clear hint we can provide for all $\langle M, w \rangle$ strings in *HALT*

Verification

```
/* imagine 5000 lines of TM code */
```

Does this code HALT on input “abbababababbbb”?

Example:

Someone gives us the number of steps hint “20.” We run the TM on the input “abbababababbbb” and observe the TM accepts the input on step 20.

This $\langle M, w \rangle$ is in the language *HALT*.

Verification

```
/* imagine 5000 lines of TM code */
```

Does this code HALT on input “abbababababbbb”?

Example:

Someone gives us the number of steps hint “20.” We run the TM on the input “abbababababbbb” and observe the TM rejects the input on step 15.

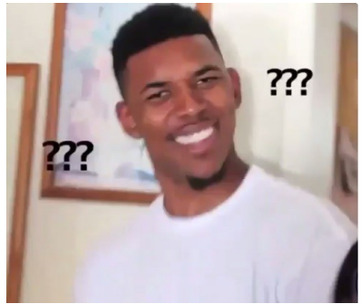
This $\langle M, w \rangle$ is in the language *HALT*.

Verification

```
/* imagine 5000 lines of TM code */
```

Does this code HALT on input “abbababababbbb”?

Is this $\langle M, w \rangle$ in the language HALT?



Example:

Someone gives us the number of steps hint “20.” We run the TM on the input “abbababababbbb” and observe the TM has neither accepted nor rejected the input after 20 steps (still running).

Is this $\langle M, w \rangle$ in the language *HALT*?

Verification

- In each of the preceding cases, we were given some problem and some evidence supporting the claim that the answer is “yes.”
- Given correct/helpful evidence, we can quickly see that the answer is indeed “yes.”
- Given incorrect/unhelpful evidence, we aren't immediately sure whether the answer is “yes.”
  - Maybe there's *no* evidence saying that the answer is “yes,” because the answer is no!
  - Or maybe there is some evidence, but just not the evidence we were given.
- Let's formalize this idea.

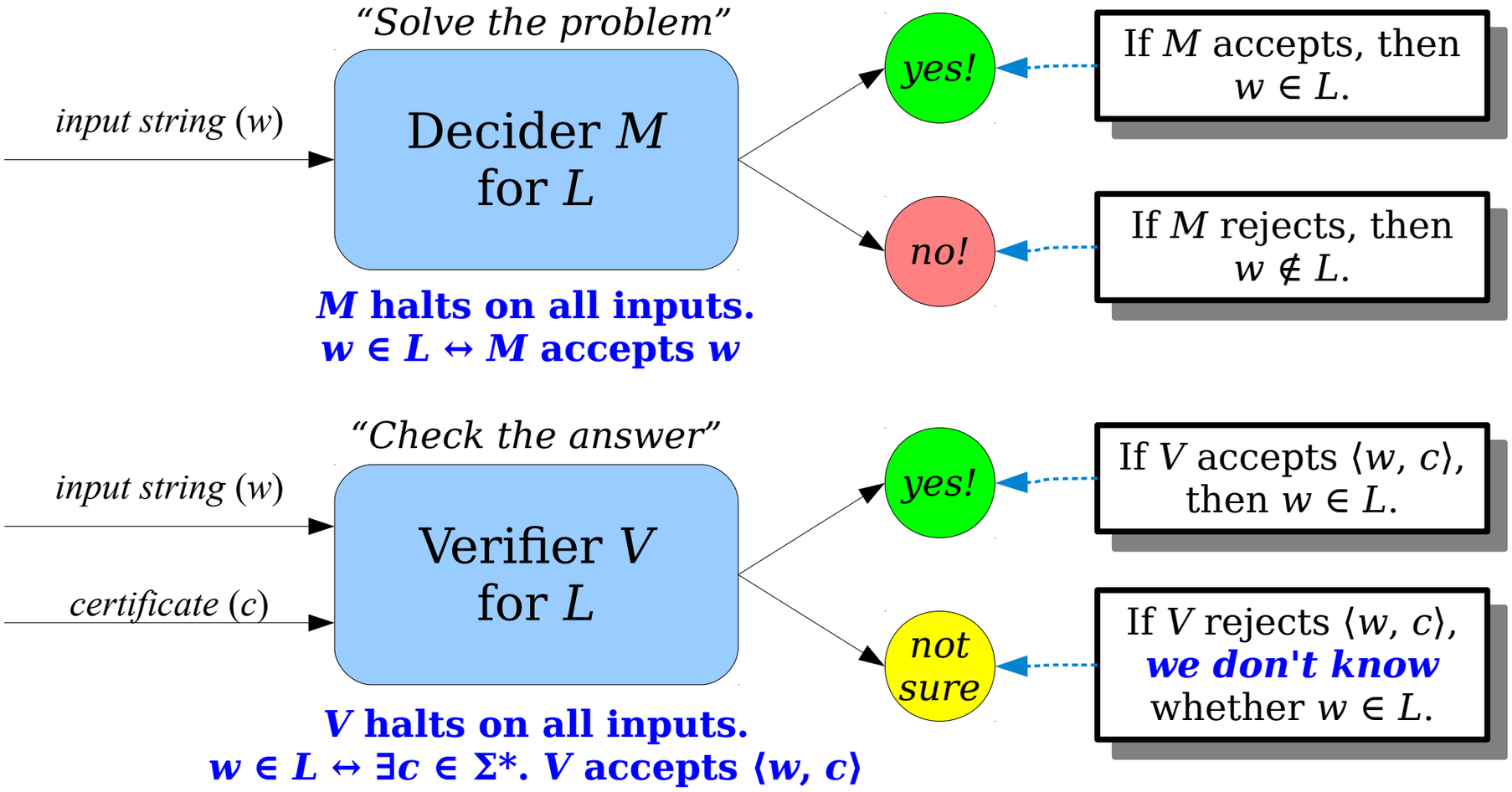
Verifiers

- A ***verifier*** for a language L is a TM V with the following properties:
  - V halts on all inputs.
  - For any string $w \in \Sigma^*$, the following is true:

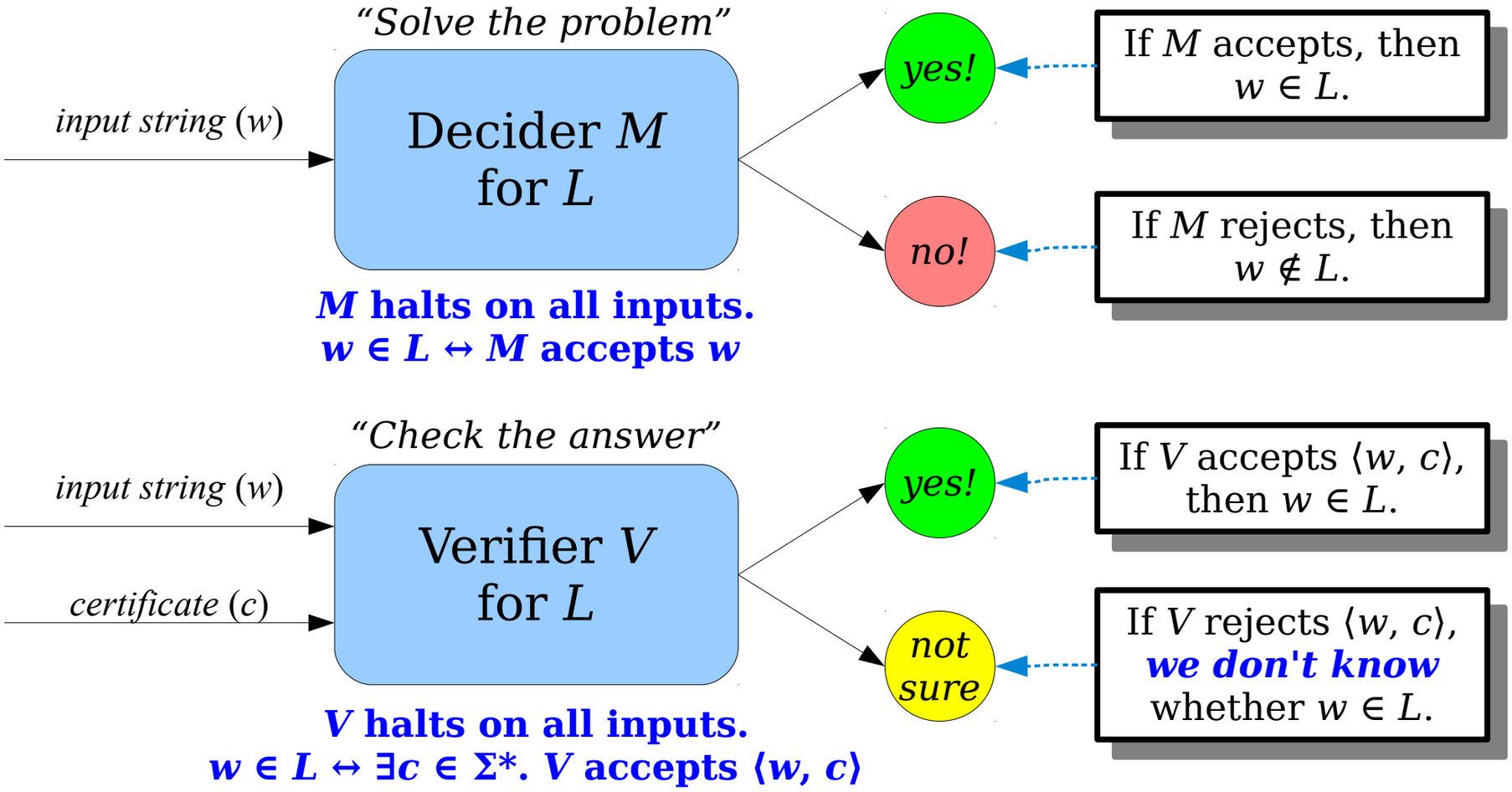
$$w \in L \iff \exists c \in \Sigma^*.\ V \text{ accepts } \langle w, c \rangle$$

- A string c where V accepts $\langle w, c \rangle$ is called a ***certificate*** for w.
  - This is the “evidence.”
- Intuitively, what does this mean?

Deciders and Verifiers

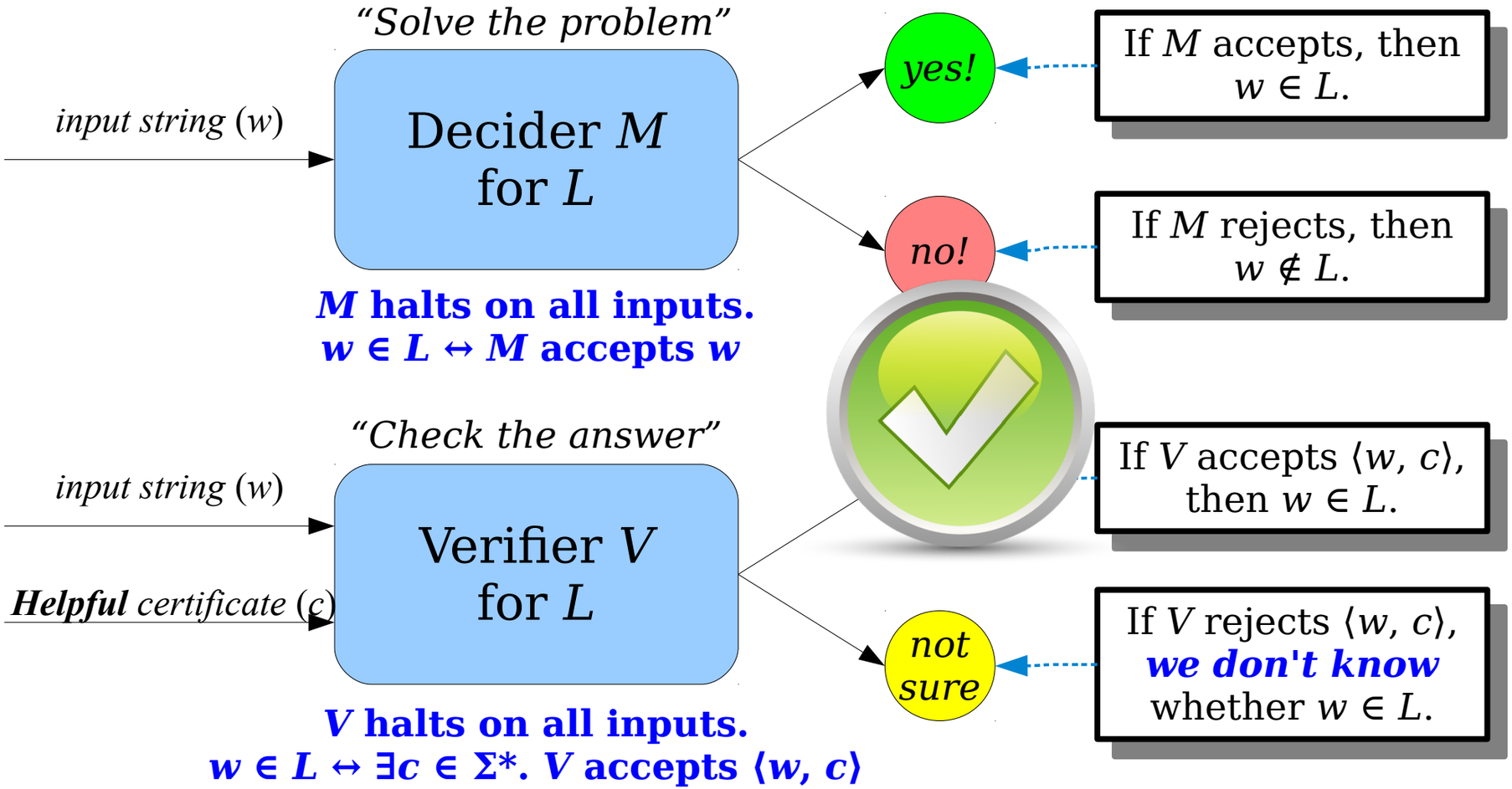


Deciders and Verifiers

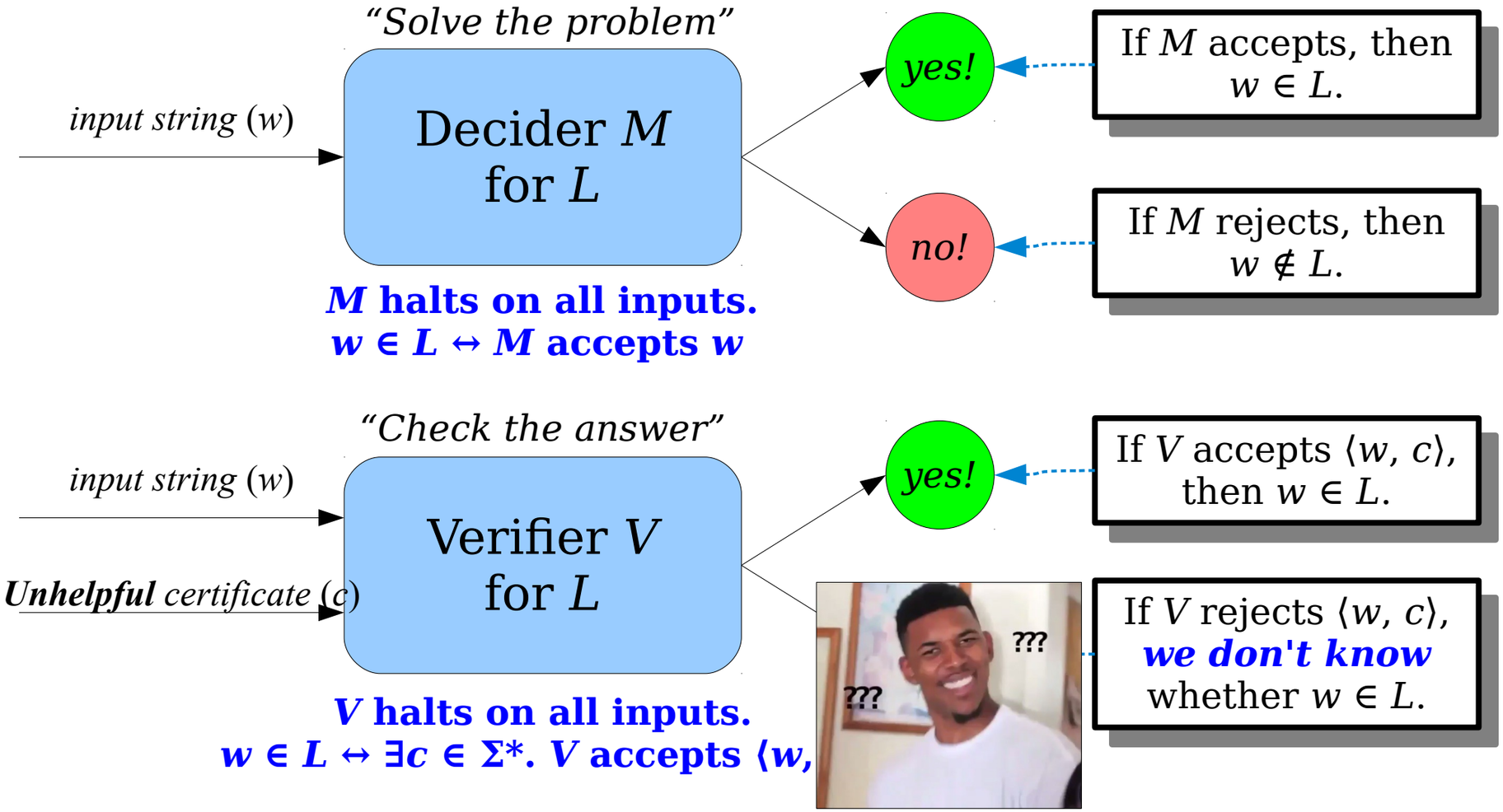


"certificate" is the official term for a "hint"

Deciders and Verifiers



Deciders and Verifiers



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  - V halts on all inputs.
  - For any string $w \in \Sigma^*$, the following is true:

$$w \in L \iff \exists c \in \Sigma^*.\ V \text{ accepts } \langle w, c \rangle$$

- Some notes about V:
  - If V accepts $\langle w, c \rangle$, then we're guaranteed $w \in L$.
  - If V does not accept $\langle w, c \rangle$, then either
    - $w \in L$, but you gave the wrong c, or
    - $w \notin L$, so no possible c will work.

Verifiers

- A ***verifier*** for a language L is a TM V with the following properties:
  - V halts on all inputs.
  - For any string $w \in \Sigma^*$, the following is true:

$$w \in L \iff \exists c \in \Sigma^*.\ V \text{ accepts } \langle w, c \rangle$$

- More notes about V:
  - Notice that c is existentially quantified.
  - Notice V is required to halt *always* (like a decider).

Verifiers

- A ***verifier*** for a language L is a TM V with the following properties:
  - V halts on all inputs.
  - For any string $w \in \Sigma^*$, the following is true:

$$w \in L \iff \exists c \in \Sigma^*.\ V \text{ accepts } \langle w, c \rangle$$

- More notes about V:
  - Notice that $\mathscr{L}(V) \neq L$. *(Good question to hold on to for a second: what <u>is</u> $\mathscr{L}(V)$?)*
  - The job of V is just to check certificates, not to decide membership in L.

Verifiers

- A ***verifier*** for a language L is a TM V with the following properties:
  - V halts on all inputs.
  - For any string $w \in \Sigma^*$, the following is true:

$$\boldsymbol{w \in L \quad \leftrightarrow \quad \exists c \in \Sigma^*.\ V \text{ accepts } \langle w, c \rangle}$$

- A note about c:
  - Figuring out what would make a good certificate (should it be a number of steps to take, an equation-solving variable assignment, a set of graph nodes, an array of numbers to fill in a whole Sudoku board?) is custom work to do for each different language L.

Some Verifiers

- Let L be the following language:

$$L = \{ \langle n \rangle \mid n \in \mathbb{N} \text{ and the hailstone sequence terminates for } n \}$$

```
bool checkHailstone(int n, int c) {
    for (int i = 0; i < c; i++) {
        if (n % 2 == 0) n /= 2;
        else n = 3*n + 1;
        if (n == 1) return true;
    }
    return n == 1;
}
```

Some Verifiers

Does this always halt?

$L = \{ \langle n \rangle \mid n \in \mathbb{N}$ and the hailstone sequence terminates for $n \}$

```
bool checkHailstone(int n, int c) {
    for (int i = 0; i < c; i++) {
        if (n % 2 == 0) n /= 2;
        else n = 3*n + 1;
        if (n == 1) return true;
    }
    return n == 1;
}
```

Some Verifiers

For one given $\langle n \rangle \in L$ (say 11), how many different values of c will work to cause the verifier to accept?

$L = \{ \langle n \rangle \mid n \in \mathbb{N}$ and the hailstone sequence terminates for $n \}$

```
bool checkHailstone(int n, int c) {
    for (int i = 0; i < c; i++) {
        if (n % 2 == 0) n /= 2;
        else n = 3*n + 1;
        if (n == 1) return true;
    }
    return n == 1;
}
```

How many of these statements are true of $\mathscr{L}(V)$?

- $\mathscr{L}(V) = L$
- $\mathscr{L}(V) \subseteq L$
- $L \subseteq \mathscr{L}(V)$

$L = \{ \langle n \rangle \mid n \in \mathbb{N}$ and the hailstone sequence terminates for $n \}$

```
bool checkHailstone(int n, int c) {
    for (int i = 0; i < c; i++) {
        if (n % 2 == 0) n /= 2;
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    }
    return n == 1;
}
```
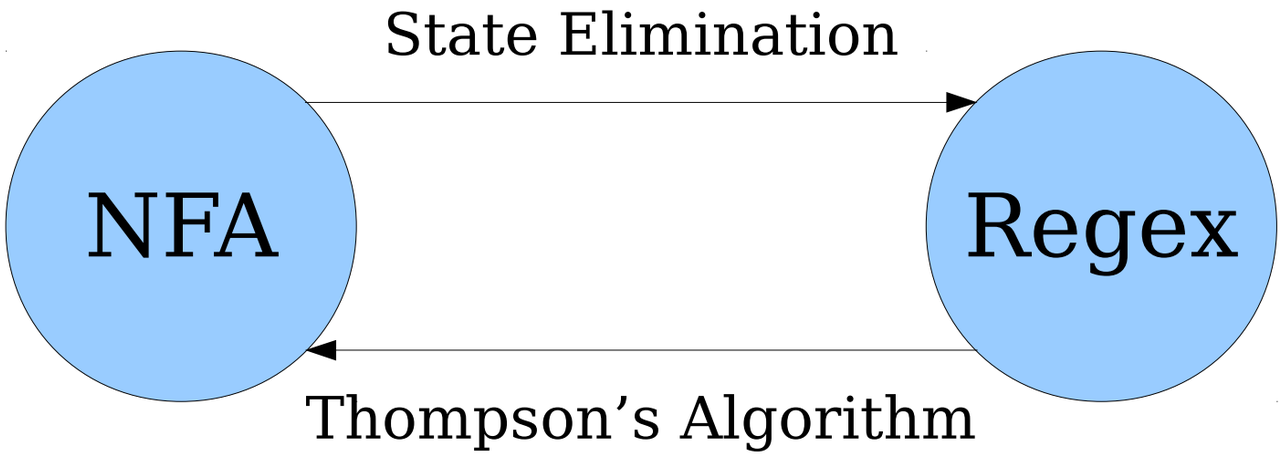
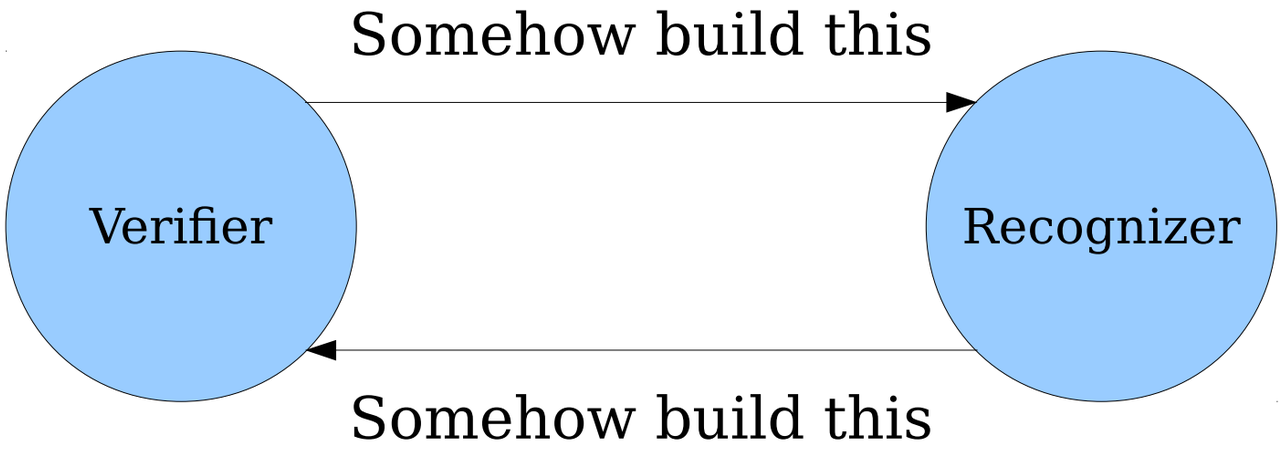

Where We've Been



Where We’re Going Today



Verifier for A_{TM}?

- Consider A_{TM}:

  $A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}$.

- This is our standard example of an undecidable language. There's no way, in general, to tell whether a TM M will accept a string w.
- Although this language is undecidable, it's an **RE** language, and it's possible to build a verifier for it!

What would make a good certificate for a verifier for A_{TM}?

- Consider A_{TM}:

  $A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}$.

- This is a ***canonical*** example of an undecidable language. There's no way, in general, to tell whether a TM M will accept a string w.
- Although this language is undecidable, it's an **RE** language, and it's possible to build a verifier for it!

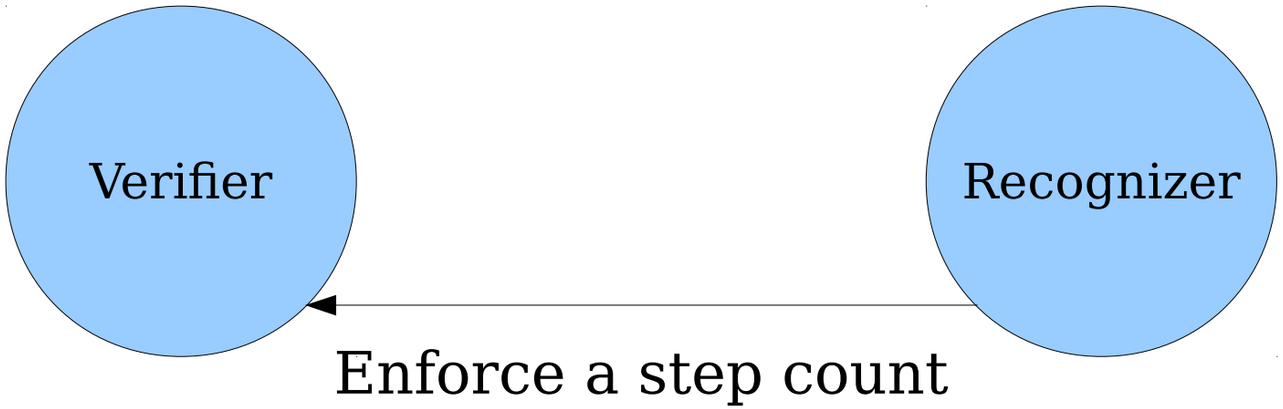
A Verifier for A_{TM}

- Recall $A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}$

```
bool checkWillAccept(TM M, string w, int c) {
    set up a simulation of M running on w;
    for (int i = 0; i < c; i++) {
       simulate the next step of M running on w;
    }
    return whether M is in an accepting state;
}
```

- Do you see why M accepts w iff there is some c such that `checkWillAccept(M, w, c)` returns true?
- Do you see why `checkWillAccept` always halts?

Equivalence of Verifiers and Recognizers



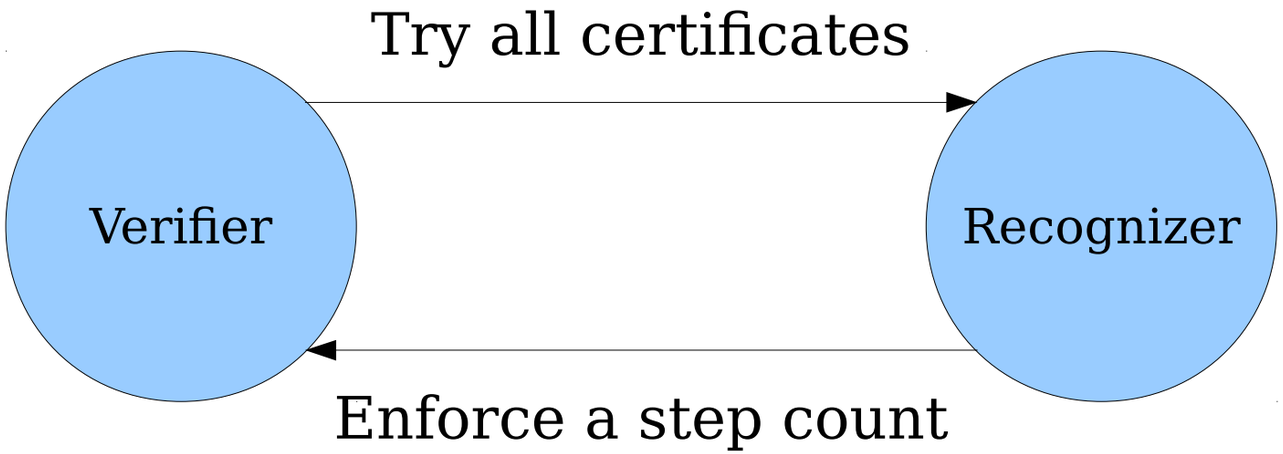
What languages are verifiable?

Let V be a verifier for a language L. Consider the following function given in pseudocode:

```
bool mysteryFunction(string w) {
    int i = 0;
    while (true) {
        for (each string c of length i) {
            if (V accepts ⟨w, c⟩) return true;
        }
        i++;
    }
}
```

What set of strings does `mysteryFunction` return **`true`** on?

Equivalence of Verifiers and Recognizers



Theorem: If L is a language, then there is a verifier for L if and only if $L \in \mathbf{RE}$.

Verifiers and **RE**

- ***Theorem:*** If there is a verifier V for a language L, then $L \in \mathbf{RE}$.
- ***Proof goal:*** Given a verifier V for a language L, find a way to construct a recognizer M for L.

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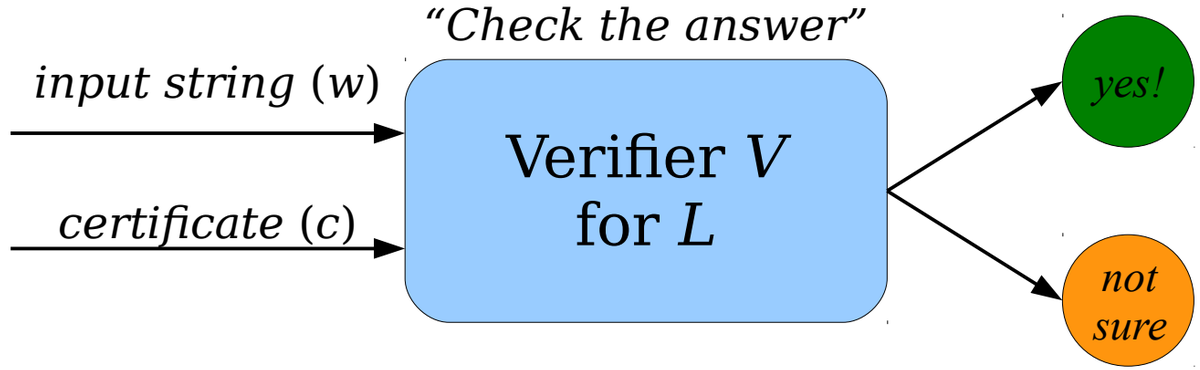

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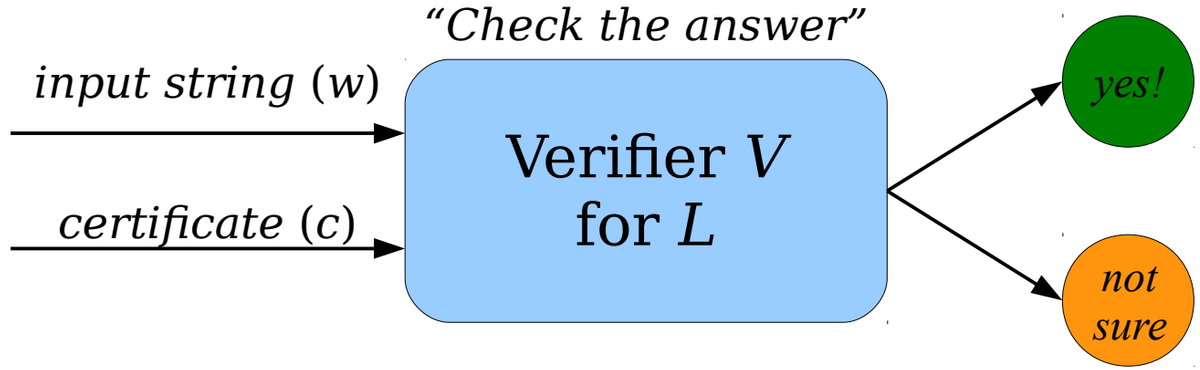
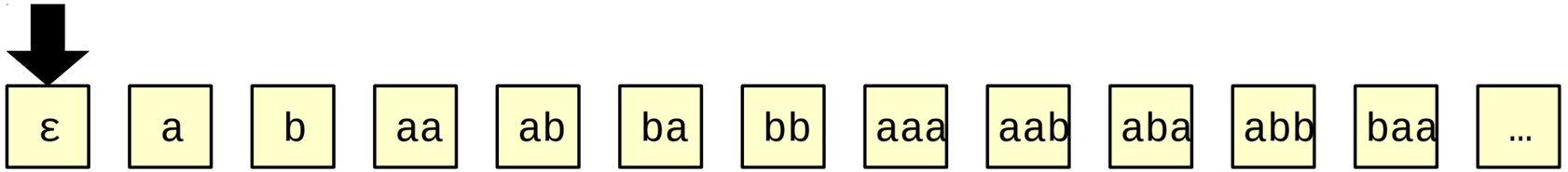


We will try all possible certificates (*values of c*)



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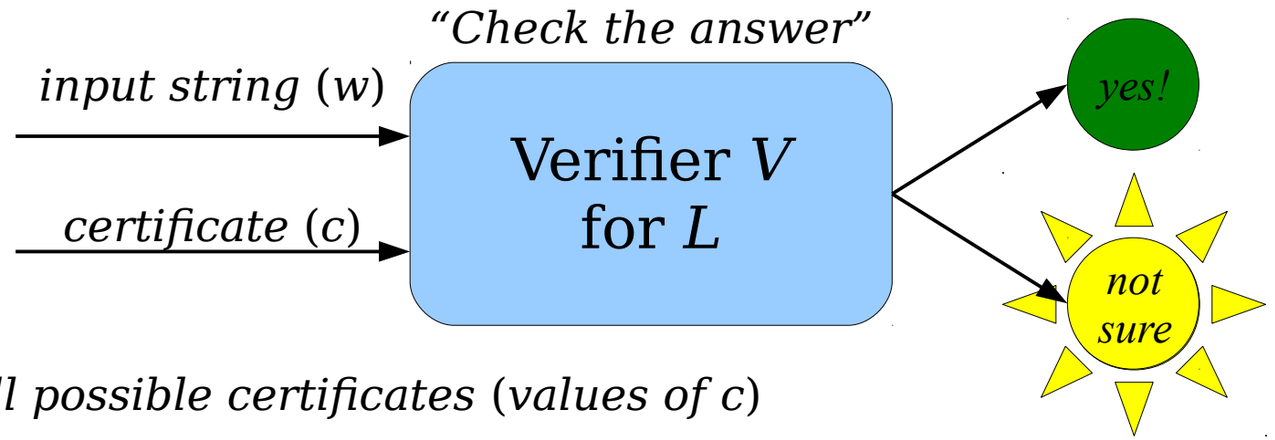

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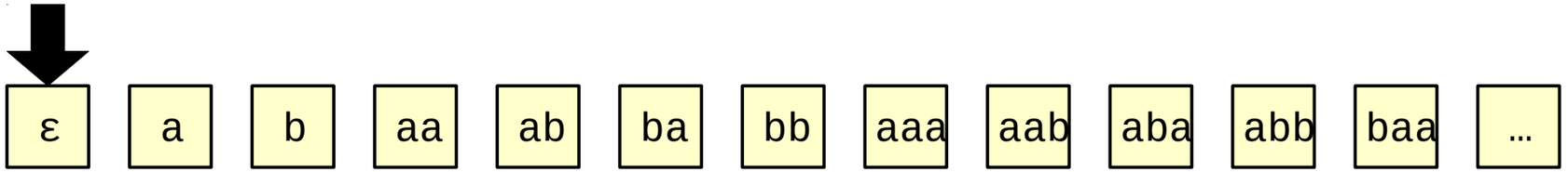

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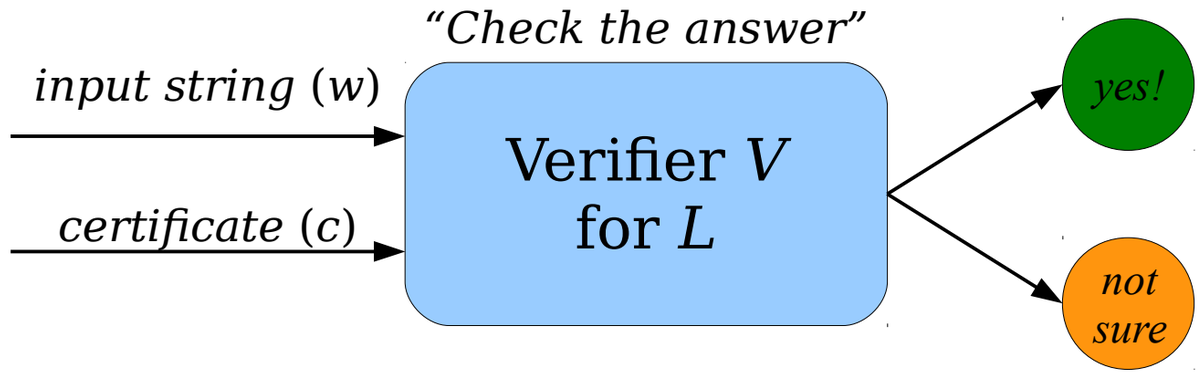

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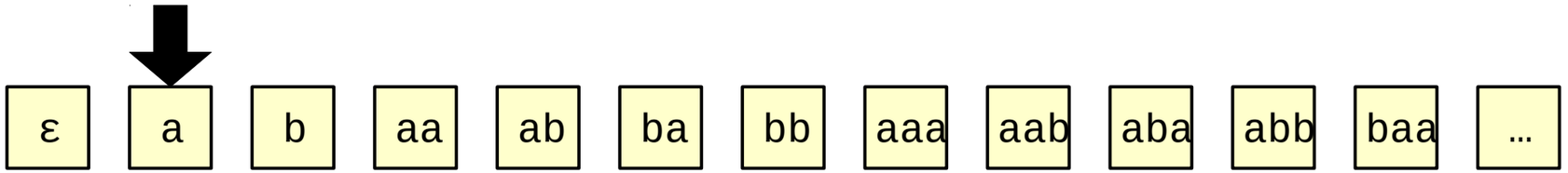

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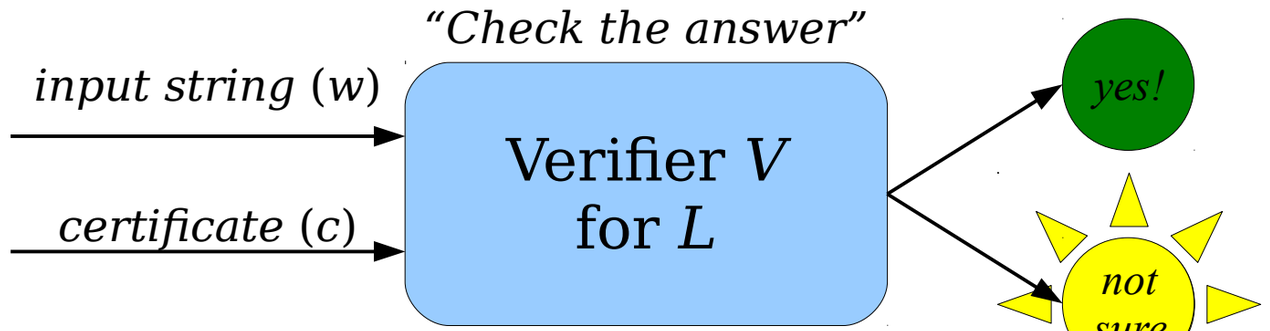

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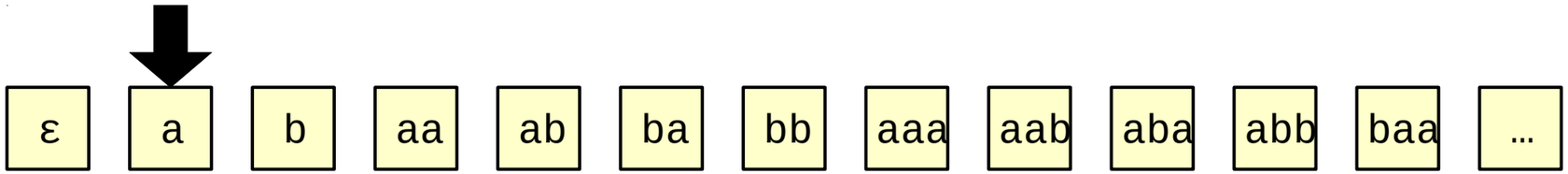

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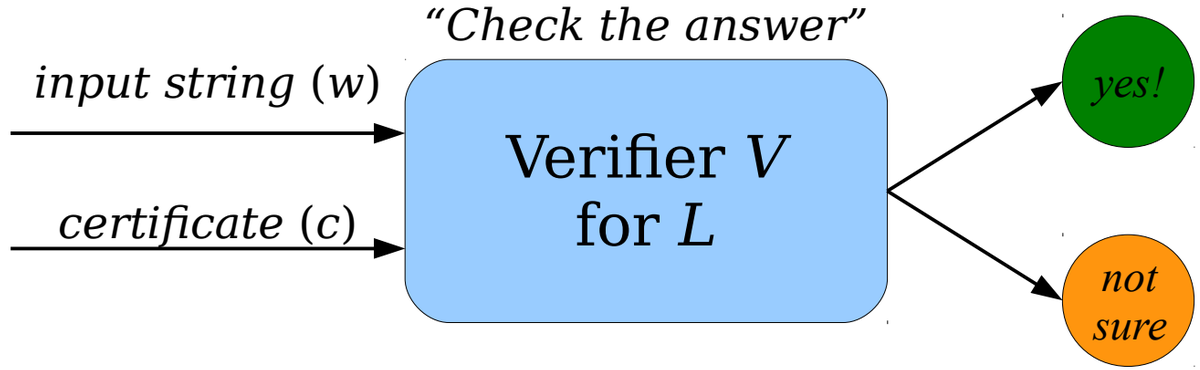

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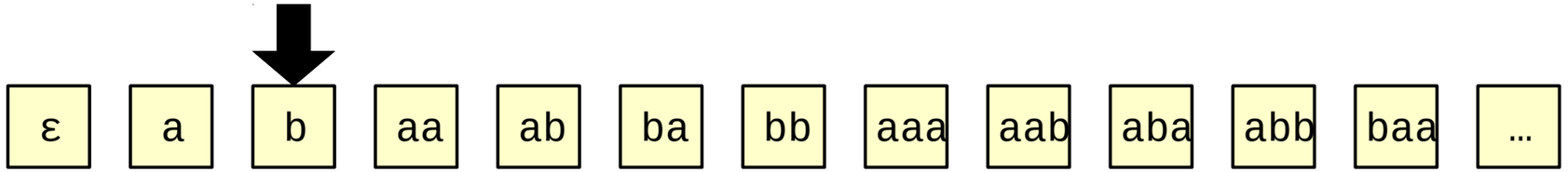

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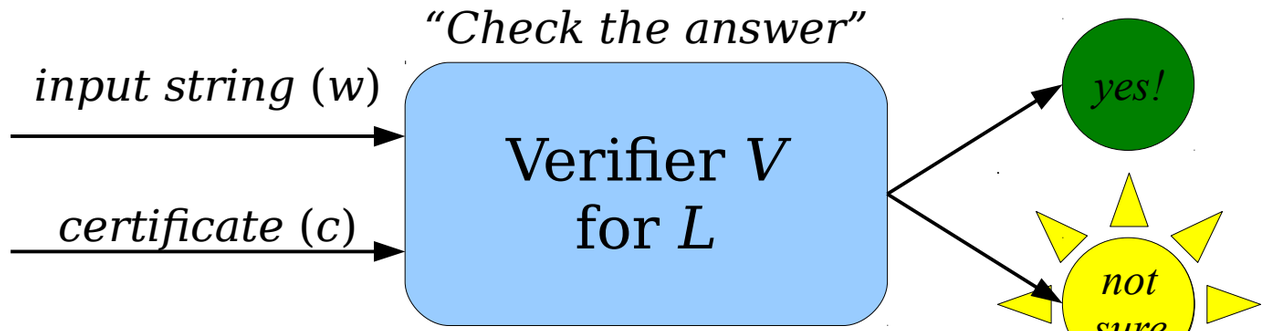

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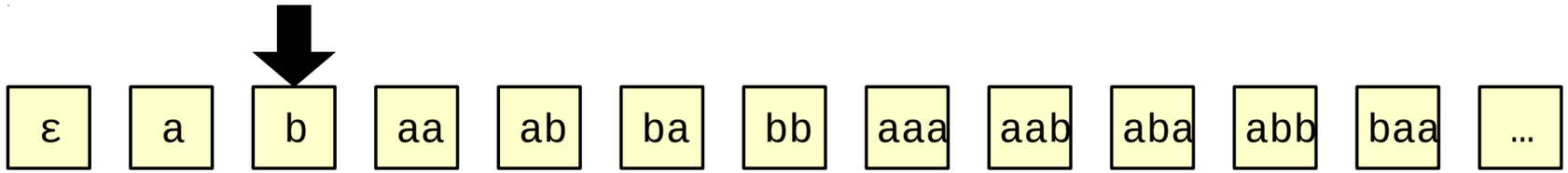

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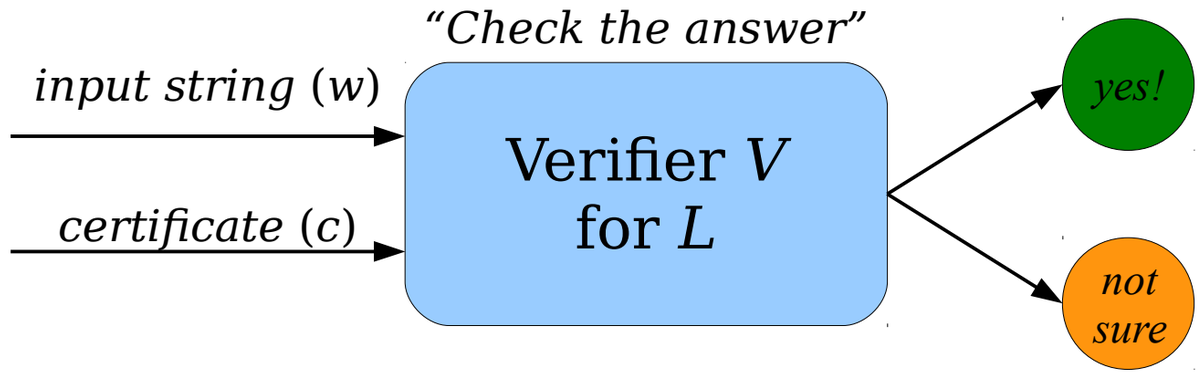

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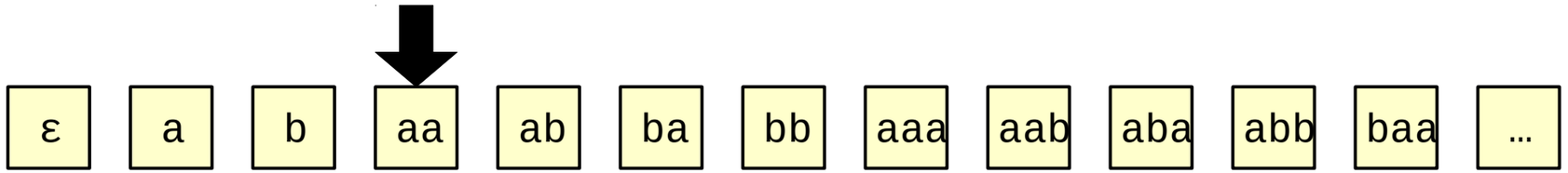

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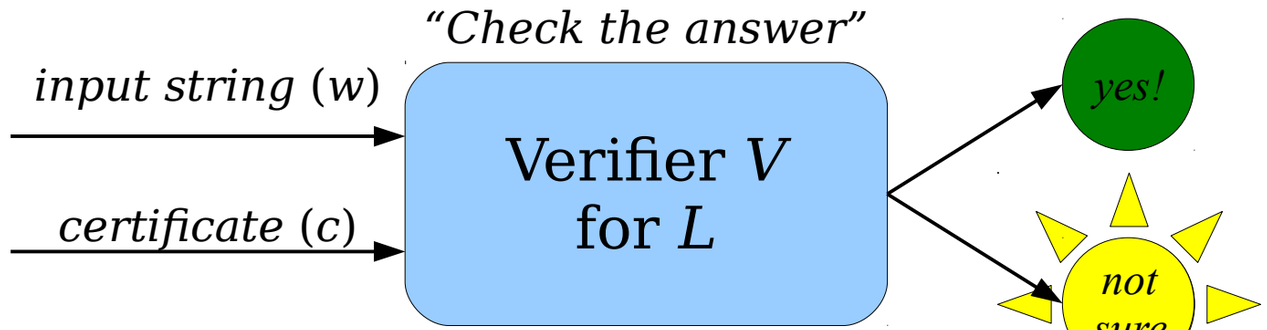

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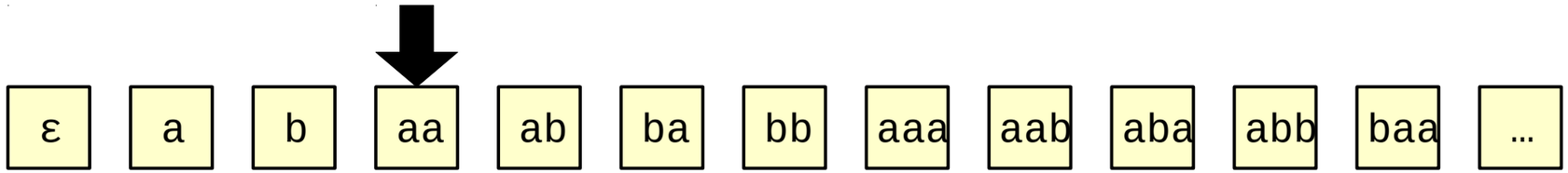

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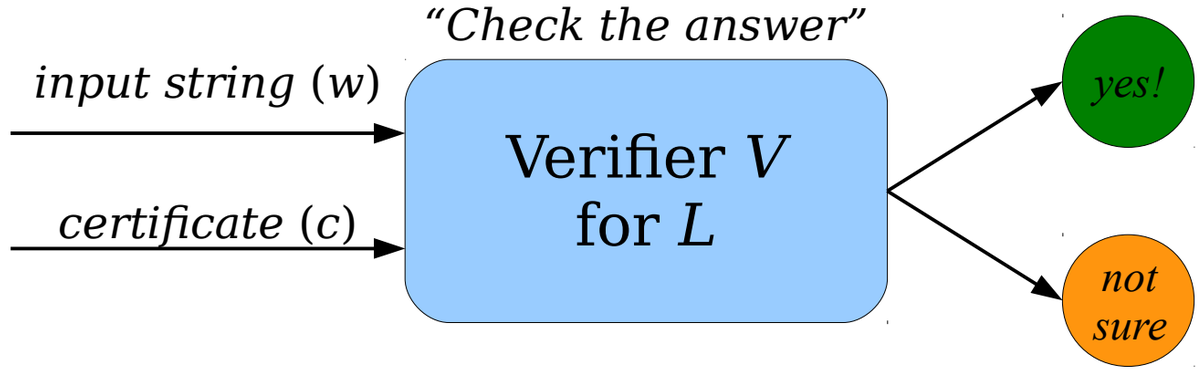

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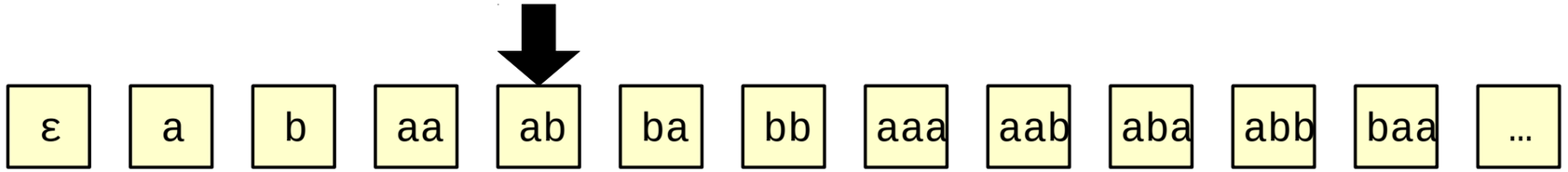

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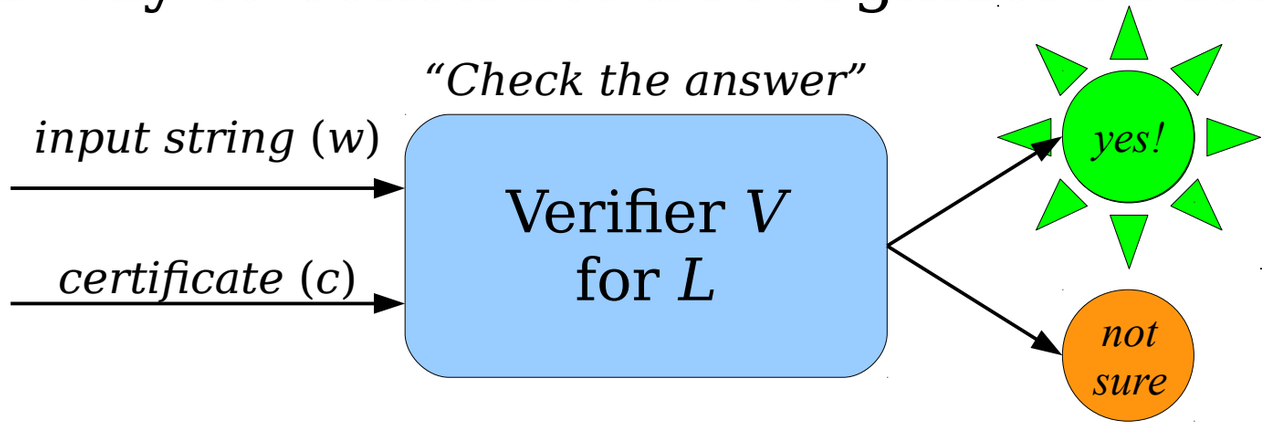

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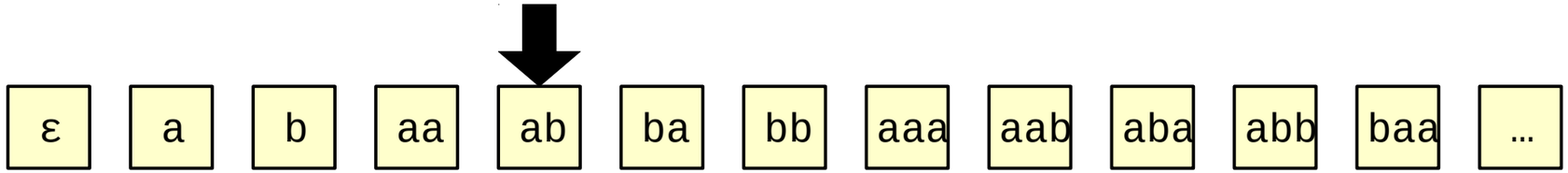

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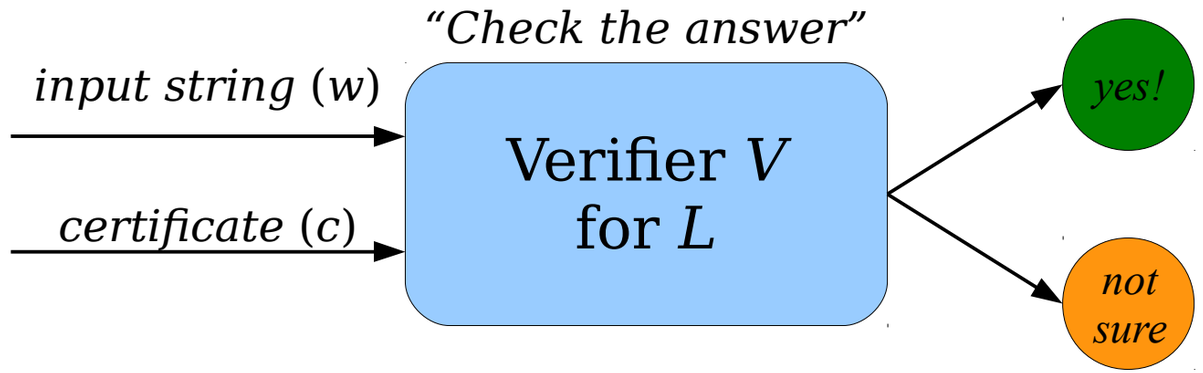

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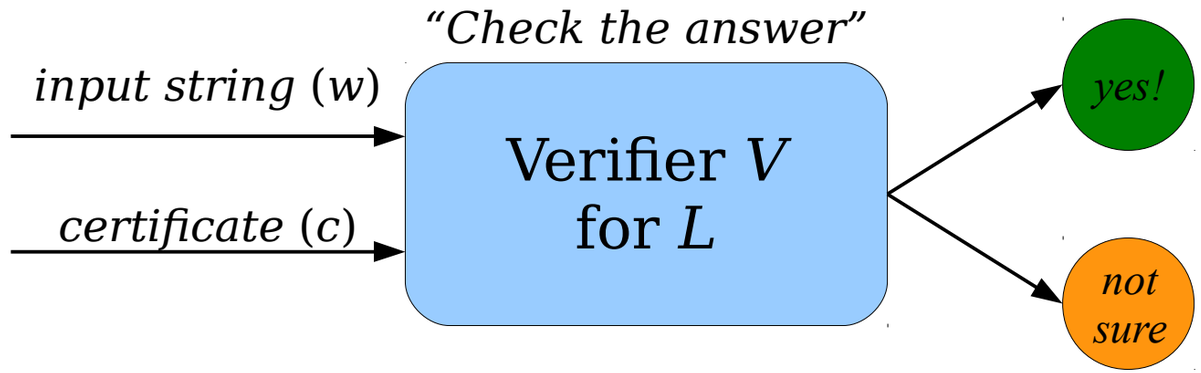

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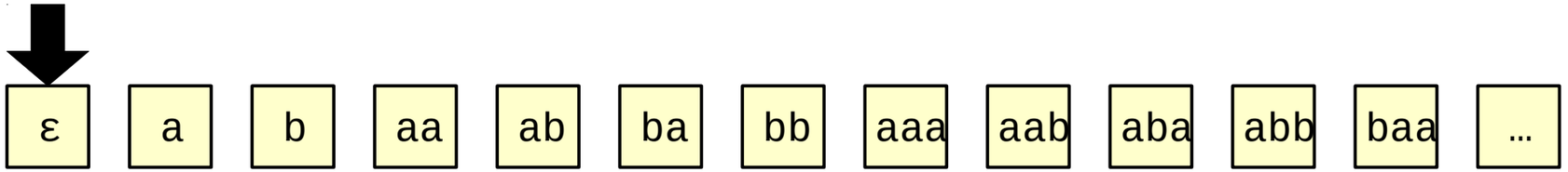

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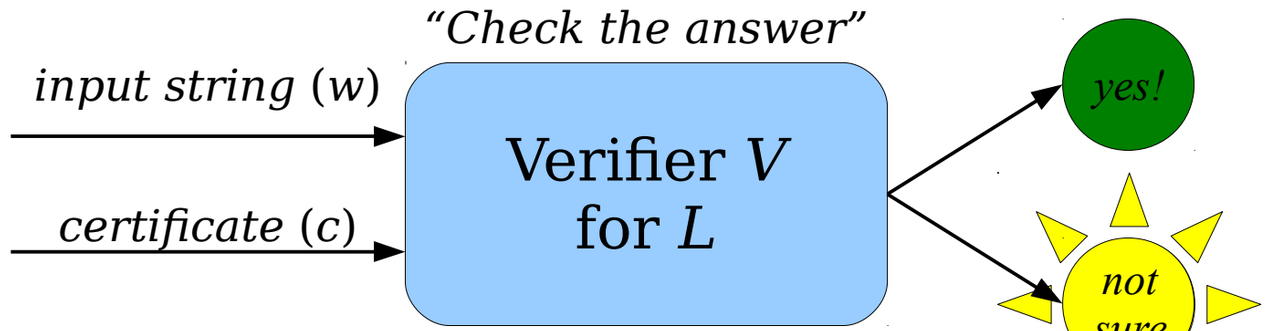

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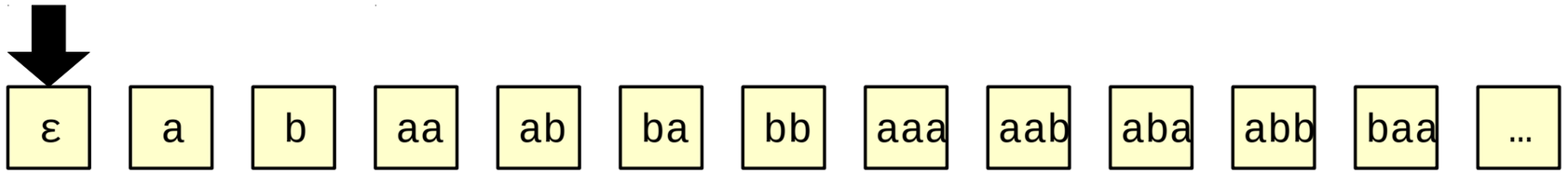

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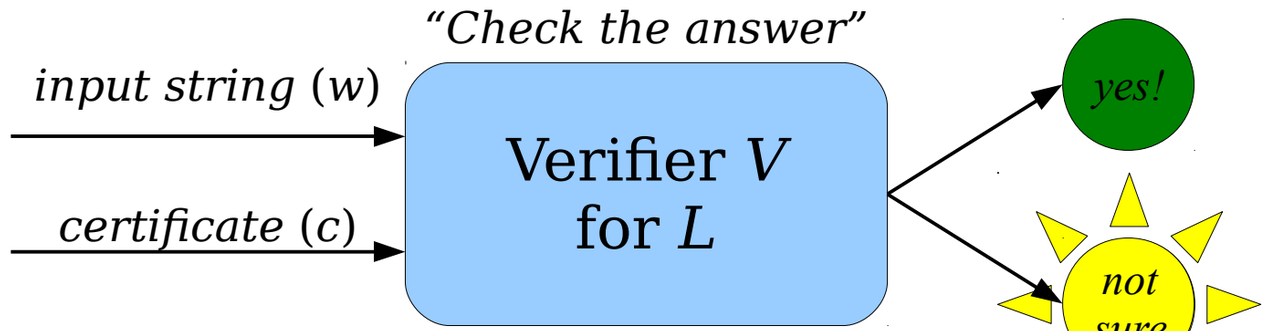

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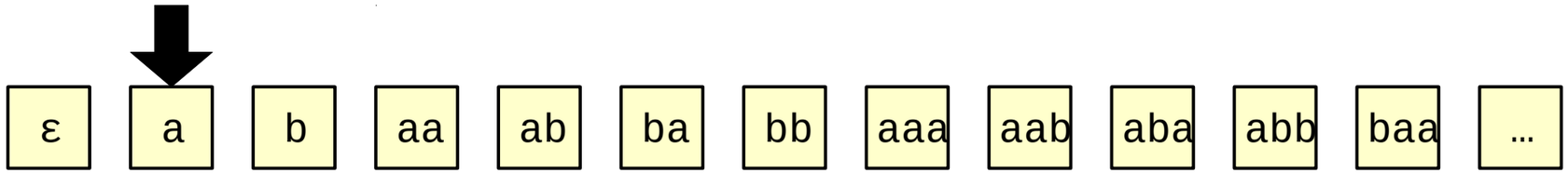

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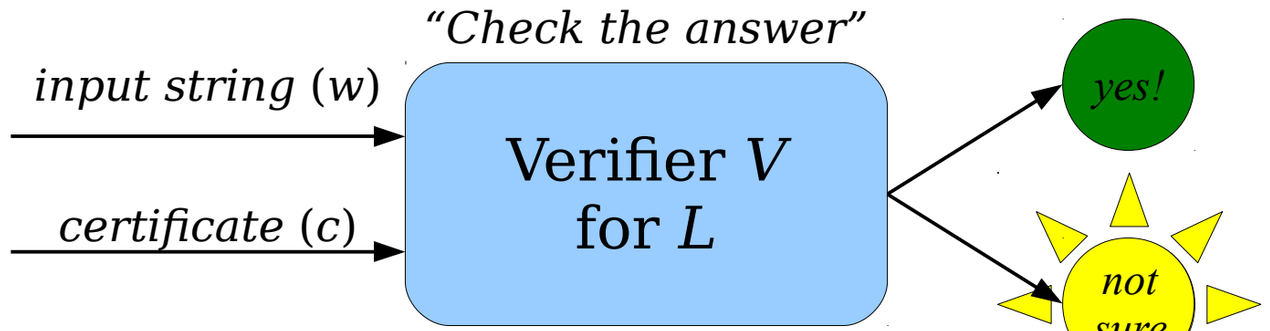

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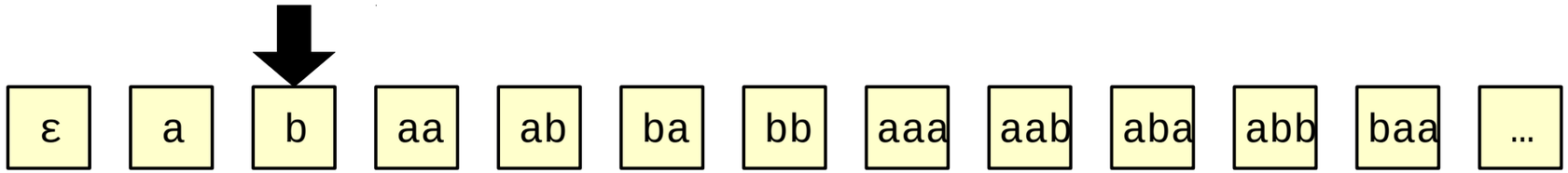

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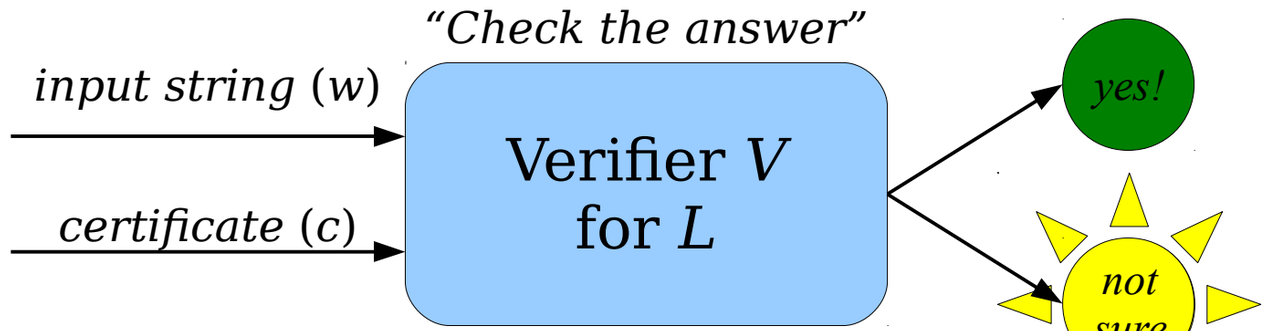

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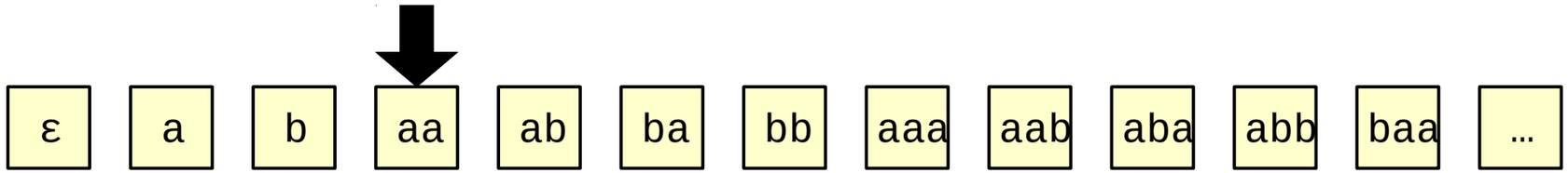

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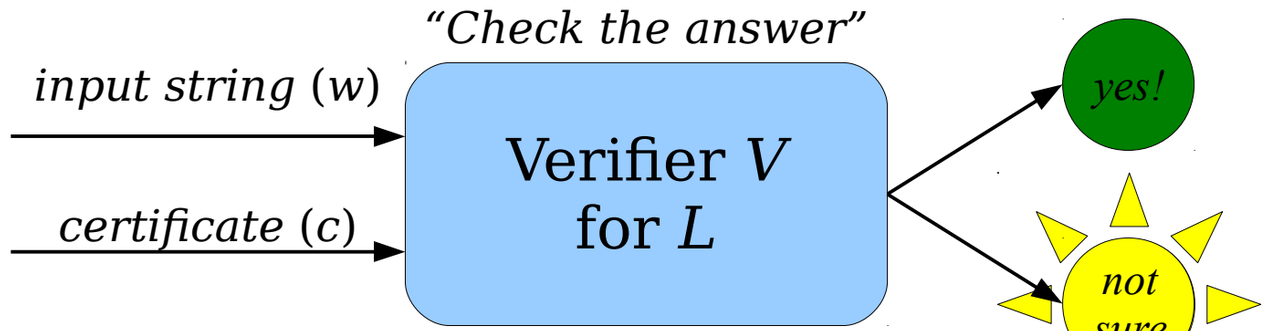

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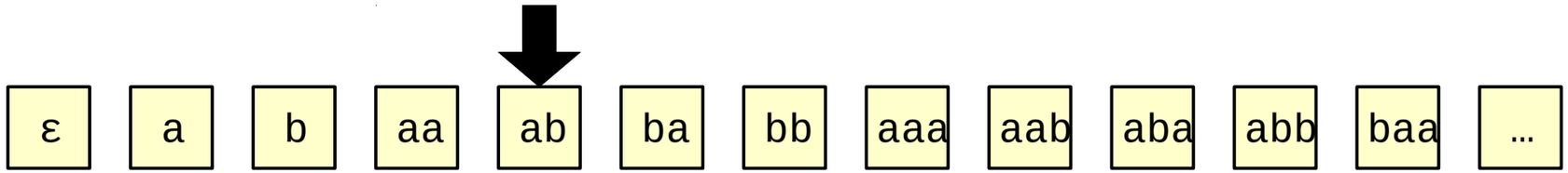

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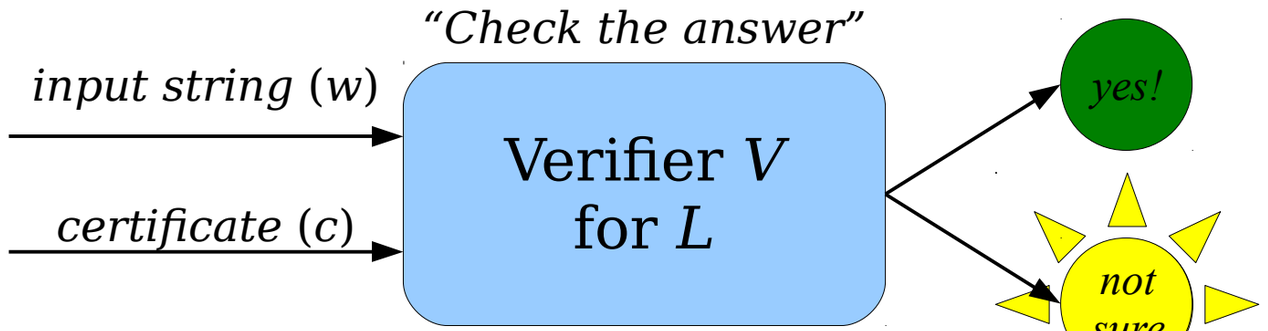

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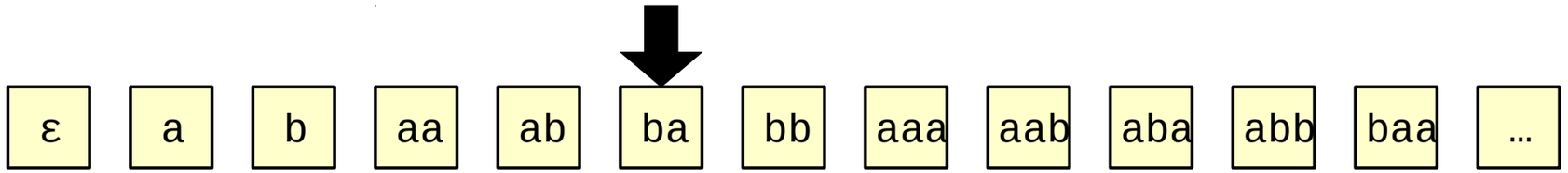

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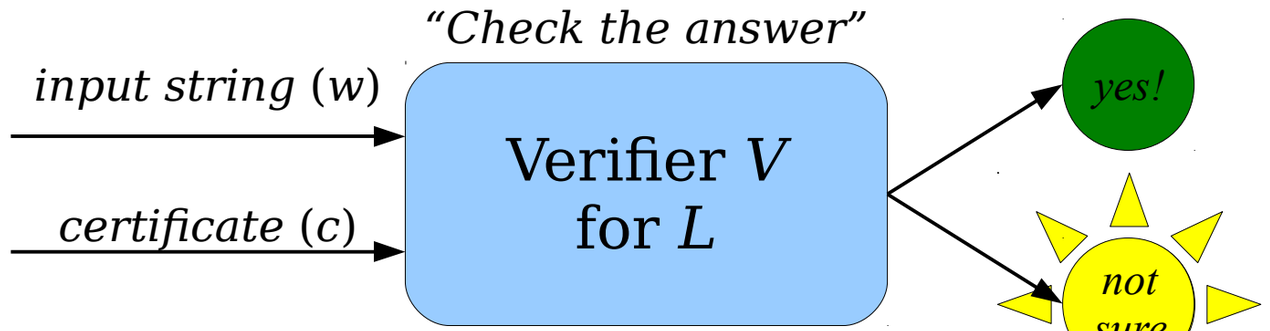

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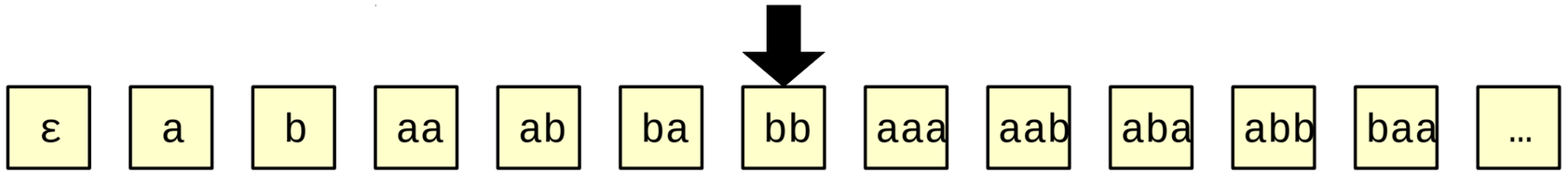

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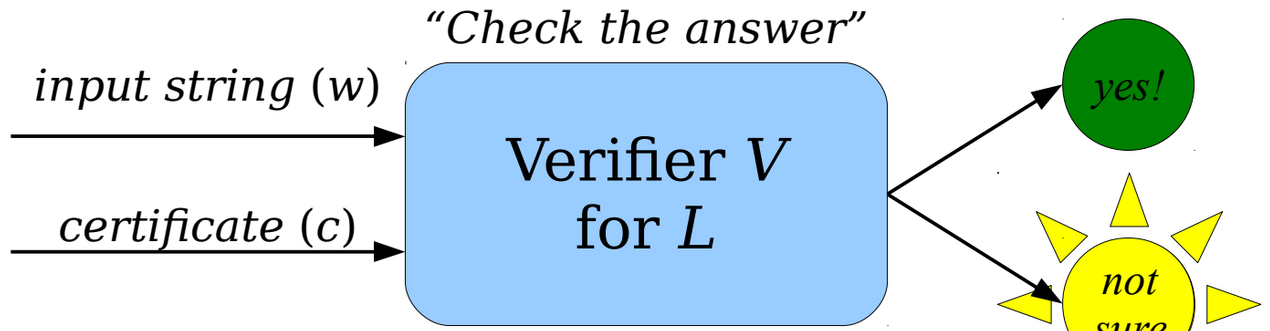

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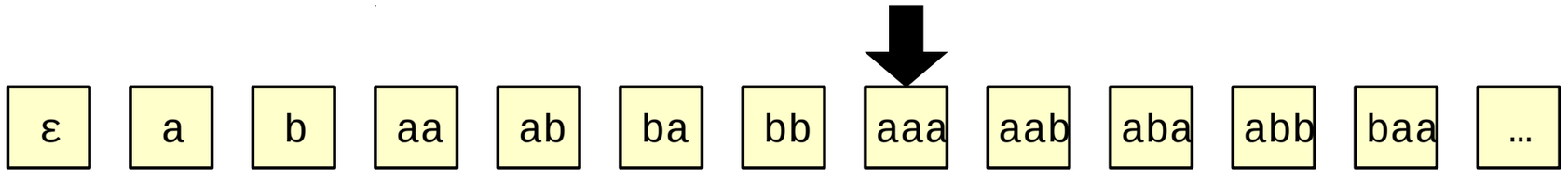

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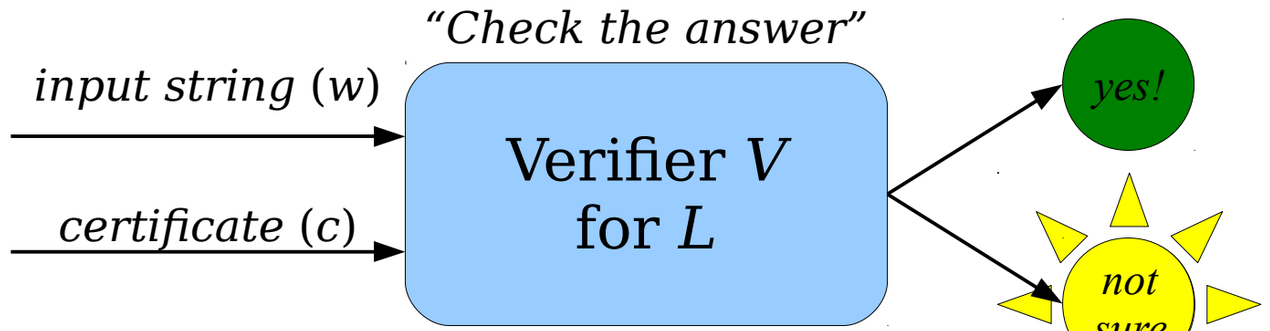

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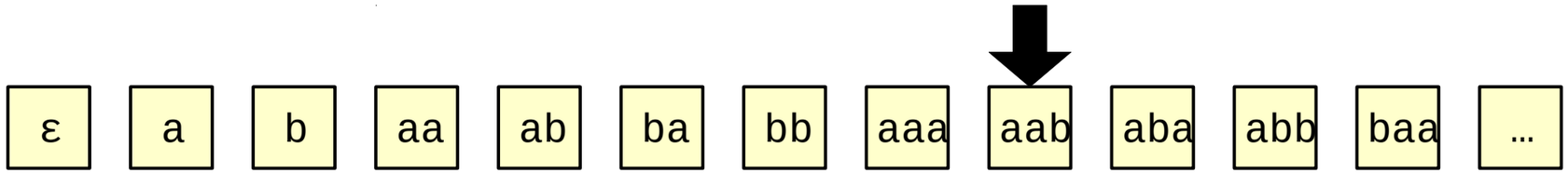

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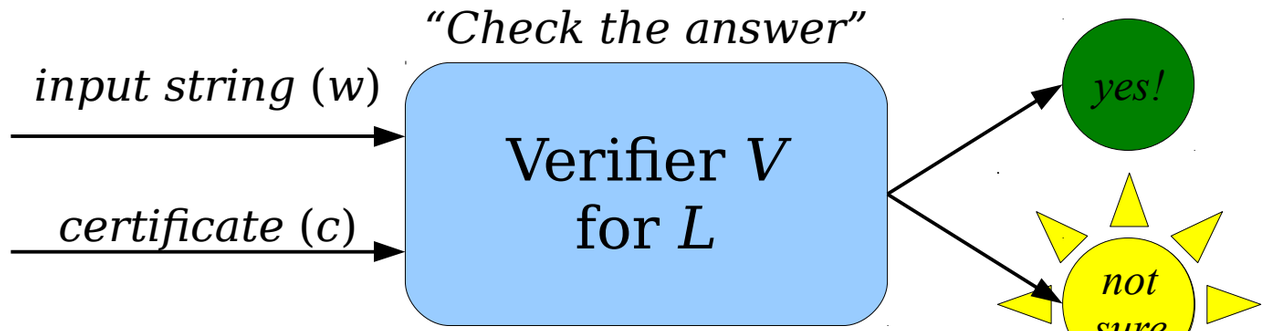

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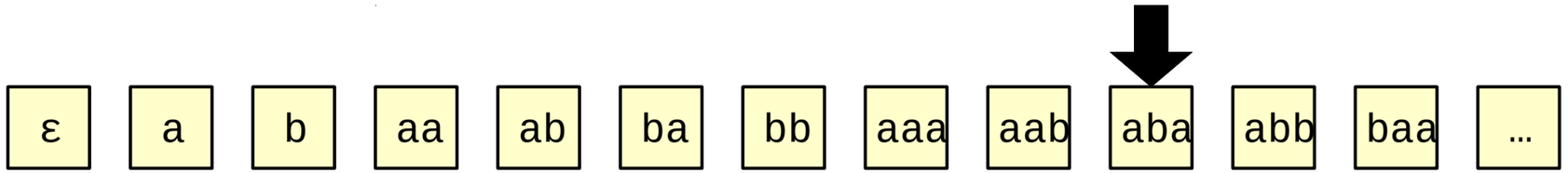

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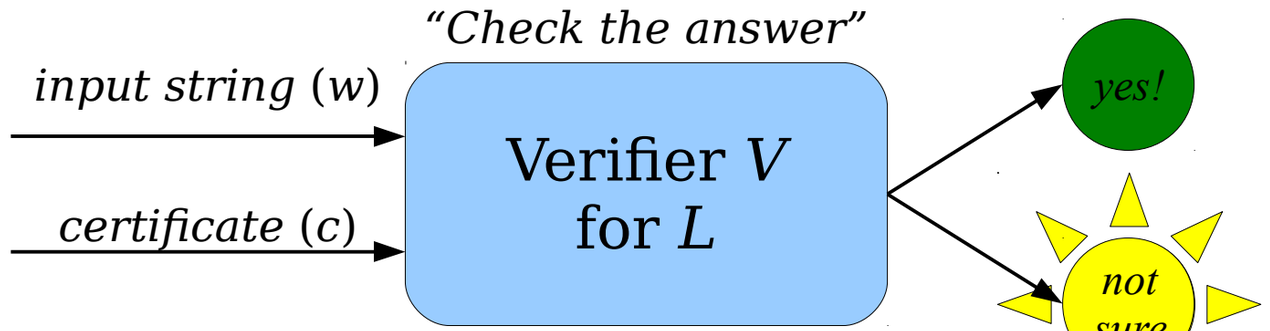

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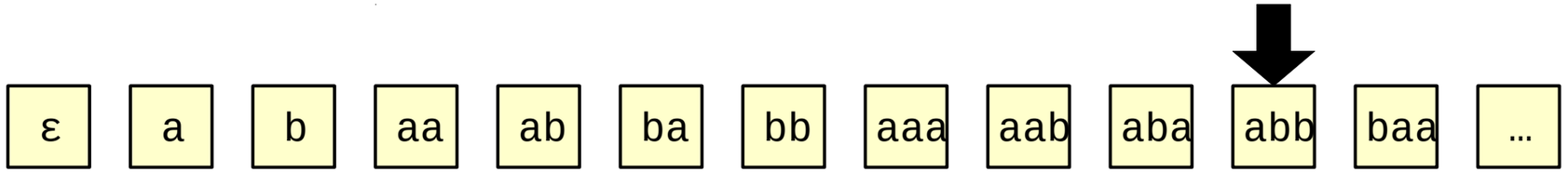

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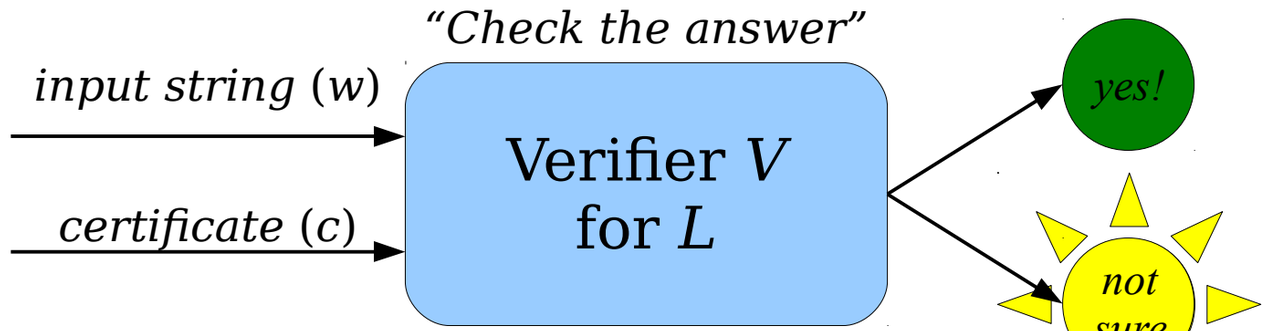

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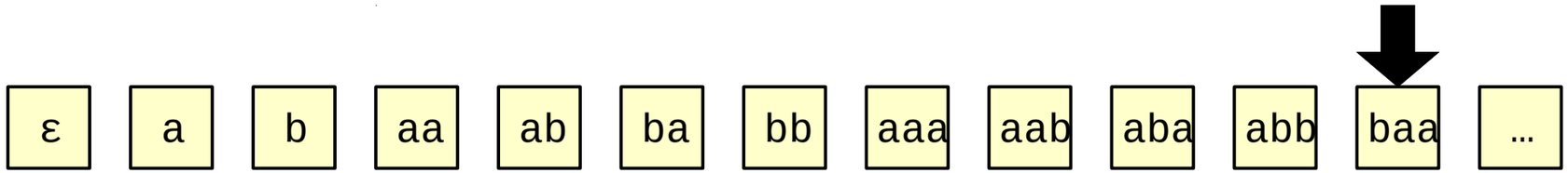

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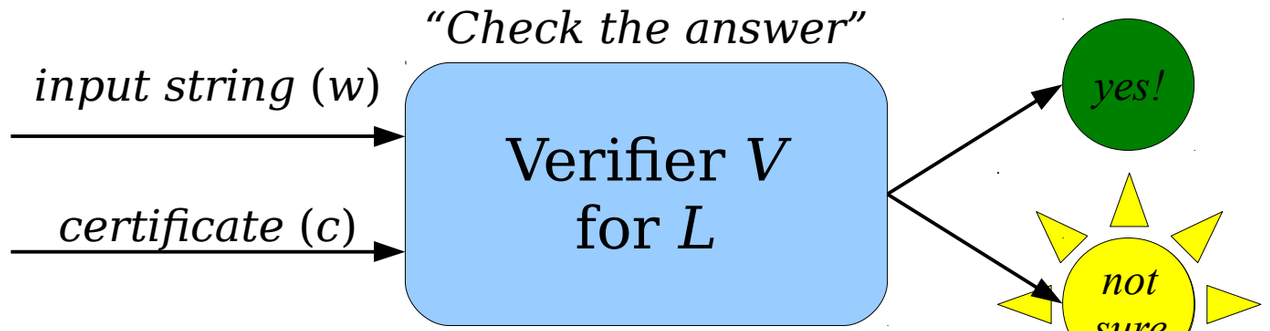

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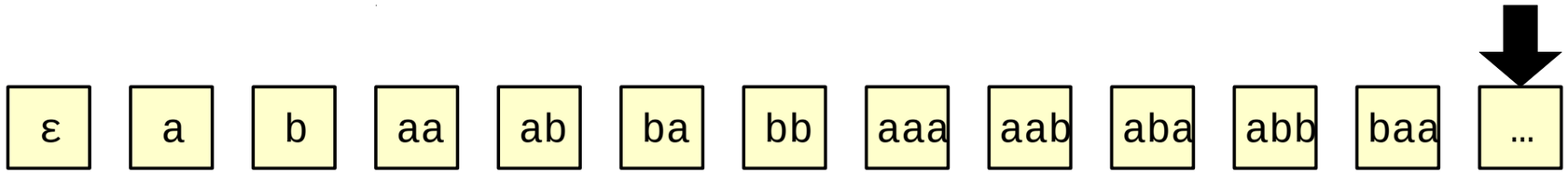

Verifiers and **RE**

- ***Theorem:*** If V is a verifier for L, then $L \in \mathbf{RE}$.
- ***Proof sketch:*** Consider the following program:

```
bool isInL(string w) {
   int i = 0;
   while (true) {
      for (each string c of length i) {
          if (V accepts ⟨w, c⟩) return true;
      }
      i++;
   }
}
```

If $w \in L$, there is some $c \in \Sigma^*$ where V accepts $\langle w, c \rangle$. The function `isInL` tries all possible strings as certificate, so it will eventually find c (or some other certificate), see V accept $\langle w, c \rangle$, then return true. Conversely, if `isInL(w)` returns true, then there was some string c such that V accepted $\langle w, c \rangle$, so $w \in L$. ■

Verifiers and **RE**

- ***Theorem:*** If $L \in$ **RE**, then there is a verifier for L.
- ***Proof goal:*** Beginning with a recognizer M for the language L, show how to construct a verifier V for L.
- The challenges:
  - A recognizer M is not required to halt on all inputs. A verifier V must always halt.
  - A recognizer M takes in one single input. A verifier V takes in two inputs.
- We'll need to find a way of reconciling these requirements.

Recall: If M is a recognizer for a language L, then M accepts w iff $w \in L$.

Key insight: If M accepts a string w, it always does so in a finite number of steps.

Idea: Adapt the verifier for A_{TM} into a more general construction that turns any recognizer into a verifier by running it for a fixed number of steps.

Verifiers and **RE**

- ***Theorem:*** If $L \in$ **RE**, then there is a verifier for L.
- ***Proof sketch:*** Consider the following program:

```
bool checkIsInL(string w, int c) {
    set up a simulation of M running on w;
    for (int i = 0; i < c; i++) {
        simulate the next step of M running on W;
    }
    return whether M is in an accepting state;
}
```

Notice that `checkIsInL` always halts, since each step takes only finite time to complete. Next, notice that if there is a c where `checkIsInL(w, c)` returns true, then M accepted w after running for c steps, so $w \in L$. Conversely, if $w \in L$, then M accepts w after some number of steps (call that number c). Then `checkIsInL(w, c)` will run M on w for c steps, watch M accept w, then return true. ■

RE and Proofs

- Verifiers and recognizers give two different perspectives on the “proof” intuition for **RE**.
- Verifiers are explicitly built to check proofs that strings are in the language.
  - If you know that some string w belongs to the language and you have the proof of it, you can convince someone else that $w \in L$.
- You can think of a recognizer as a device that “searches” for a proof that $w \in L$.
  - If it finds it, great!
  - If not, it might loop forever.

RE and Proofs

- If the **RE** languages represent languages where membership can be proven, what does a non-**RE** language look like?
- Intuitively, a language is *not* in **RE** if there is no general way to prove that a given string $w \in L$ actually belongs to L.
- In other words, even if you knew that a string was in the language, you may never be able to convince anyone of it!